# A new condition for finite time boundedness analysis

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**Abstract**

In this paper, a new condition for finite time boundedness analysis is presented. Firstly, a brief discussion of the available conditions found in the literature and comparisons between them regarding computational efficiency and conservatism are presented. Then, a new condition expressed in terms of linear matrix inequalities (LMIs) is derived using the Finsler's lemma. The proposed condition is proved to be less conservative than LMI conditions previously presented in the literature, and its efficiency is illustrated with numerical examples.

*Keywords:* finite time stability, finite time boundedness, linear matrix inequality.

## 1. Introduction

Most systems faced in engineering do not need to operate over a long period of time, but only during a predetermined time period. This fact motivated, as early as in the 1960 decade, the introduction of the nowadays well established concepts of finite time boundedness and stability [1, 2, 3, 4, 5]. Formally, a time-varying



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linear system subject to a disturbance $\omega$ in a prespecified class $\mathscr{W}$ described by

$$\dot{x}(t) = A(t)x(t) + B(t)\omega(t),\ \forall t \in [0, T_f] \tag{1}$$

is said to be **finite time bounded** (FTB) with respect to $(c_1, c_2, T_f, R, \mathscr{W})$ with $c_2 > c_1 \geq 0$, $T_f > 0$ and $R \succ 0$ if

$$x^T(0)Rx(0) < c_1 \Rightarrow x^T(t)Rx(t) < c_2,\ \forall t \in [0, T_f], \forall w \in \mathscr{W},$$

that is, the system is FTB if for any initial condition $x(0)$ in a given bounded region, it is guaranteed that for a certain time interval of lenght $T_f$, the state $x(t)$ remains in a (possibly larger) bounded region. Some disturbance sets $\mathscr{W}$ frequently used in the literature are the class of square integrable disturbances [6, 7, 8]

$$\mathscr{W}_d^2 := \left\{ \omega(\cdot) \middle| \int_0^{T_f} \omega^T(\tau)\,\omega(\tau)\,d\tau \leq d \right\},$$

and the class of worst case disturbances [9, 10, 11]

$$\mathscr{W}_d^\infty := \left\{ \omega(\cdot) \middle| \omega^T(t)\,\omega(t) \leq d\ \forall t \in \left[0, T_f\right] \right\}.$$

In the particular case without disturbances (i.e. $\mathscr{W} = \emptyset$) or when the system does not has inputs (i.e. $B = 0$), the definition of FTB reduces to the definition of **finite time stability** (FTS).

The FTS and FTB characterization provides a good framework to analyze system trajectories during transient time and to design control strategies able to avoid excitation of nonlinear dynamics, such as saturation [12]. Systems whose state trajectory are required to be constrained during a finite time horizon are suitable to this formulation. Real word applications include ATM networks [13], car suspension system [5], missile systems, chemical processes and airplane maneuvers [3], among others.

Although it may seem a slight modification of the corresponding infinite time concepts, the characterization of time finiteness turns to be turns into a very difficult problem. In fact, only just recently a necessary and sufficient condition for these properties was obtained in terms of differential matrix inequalities (DMI) [14, 10, 6, 15, 16, 12, 17]. Among them, in [17] is given a necessary and sufficient condition based on DMI for the system be FTS. Assuming zero initial conditions, necessary and sufficient conditions in terms of an infinite number of linear DMIs for the system be FTB with respect to the class of noise $\mathscr{W}^2_{d^2}$ are given in [10]. In the general case where the initial state of the system is non-zero, only a sufficient DMI condition for FTB is presented. In [18] is presented a sufficient DMI condition for the system be FTB with respect to the class of noise $\mathscr{W}^\infty_d$.

Except for small systems, the DMI characterization of finite time boundedness or stability is computationally intractable. Following [19], the feasibility of DMIs can be shown to be a NP-hard problem. In order to lessen the computational burden, relaxations of these DMIs have been proposed, that although more conservative, are computationally tractable. An alternative necessary and sufficient FTS condition based on the solution of a differential Lyapunov equation (DLE) is proposed in [17]. For the general FTB case, sufficient conditions based only on LMIs, which is a convex problem [20], have also been proposed in [9, 7, 21]. Other works worth mentioning in the FTB context are: [22] which deals with $\mathscr{H}_\infty$ control for norm bounded disturbances, [23] which deals with time-varying bounded disturbances for discrete time systems and, [7] which also deals with norm-bounded disturbance and whose Lyapunov function motivates the main theorem of this paper. For a more detailed discussion, see [24, 5].

In this paper, the goal is to provide a computationally tractable condition for

FTB (in this sense better than DMI conditions) and also reduce the conservatism of LMI conditions presented in the literature. For this end, it is proposed to apply Finsler's lemma [25] in order to get advantage of introduced extra variables.

This paper is organized as follows. A review of the main results on FTB and FTS are presented in Section 2. A new LMI condition for FTB based on Finsler's lemma is derived in Section 3. Numerical examples are shown in Section 4 and finally, the conclusion is presented in Section 5.

In the sequel, the following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{m\times n}$ the set of real matrices of order $m\times n$, $\mathbb{S}^n$ the set of symmetric real matrices of order $n\times n$, $\mathbb{S}^n_+$ the set of symmetric positive definite matrices of order $n\times n$. $P\succ 0$ indicates that $P$ is a symmetric positive definite matrix. The symbol $\star$ indicates symmetric terms in symmetric matrices. $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are respectively the maximum and the minimum eigenvalue of the symmetric matrix $A$. $A^T$ is the transpose of matrix $A$. $\mathrm{Im}(A)$ and $\mathrm{Ker}(A)$ are respectively the image and the kernel of the matrix $A$.

## 2. Preliminary results and problem statement

In [10], finite time boundedness property with respect to the class of noise $\mathscr{W}^2_{d^2}$ for a system with zero initial state is given as the following necessary and sufficient DMI conditions.

**Lemma 1.** *System (1) with $x(0)=0$ is FTB with respect to the parameters $(c_1,c_2,$ $T_f,R,\mathscr{W}^2_{d^2})$ if and only if there exists a continuously differentiable piecewise function $P:\mathbb{R}\to\mathbb{S}^n$ such that for each $t\in[0,T_f]$ one has that*

$$\dot{P}(\tau)+A^T(\tau)P(\tau)+P(\tau)A(\tau)+\hat{\gamma}P(\tau)B(\tau)B^T(\tau)P(\tau)\prec 0,\ \tau\in[0,t],\quad (2)$$

$$P(t) \succeq R,$$

$$P(0) \prec \hat{\gamma} R,$$

*where* $\hat{\gamma} c_2/(c_1+d)$.

Lemma 1 is very hard to be of practical use: firstly, the condition (2) requires to check an infinite number of differential matrix inequalities (one for each $t \in [0,T_f]$); secondly, the strong hypothesis of zero initial condition is required. Thus, it is required a trade-off between the practical usefulness and the conservatism of the FTB/FTS conditions. In fact, [6] and [18] are presented weaker conditions in terms of one DMI for the FTB analysis of a system with possibly non-zero initial state. These sufficient (but not necessary) DMI conditions are summarized in the following Lemma 2. For the case of $\mathscr{W} = \mathscr{W}_d^2$, the condition was originally presented in [6], and for the case of $\mathscr{W} = \mathscr{W}_d^\infty$, in [18].

**Lemma 2.** *System (1) is FTB with respect to the parameters* $(c_1, c_2, T_f, R, \mathscr{W}_d^2)$ *(or with respect to* $(c_1, c_2, T_f, R, \mathscr{W}_d^\infty)$*) if there exists a continuously differentiable piecewise function* $P : \mathbb{R} \to \mathbb{S}^n$ *such that for each* $t \in [0,T_f]$ *one has that*

$$\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + \hat{\gamma} P(t)B(t)B^T(t)P(t) \prec 0, \tag{3}$$

$$P(t) \succeq R, \tag{4}$$

$$P(0) \prec \hat{\gamma} R, \tag{5}$$

*where* $\hat{\gamma} c_2/(c_1+d)$ *(or* $\hat{\gamma} c_2/(c_1+dT_f)$*, respectively).*

A sufficient condition for FTS can be derived as a special case of the FTB condition of Lemma 2 when $B = 0$ and $d = 0$. It is important to note that this analysis condition has also been shown to be necessary for finite time stability in [17]. These results are summarized in Lemma 3.

**Lemma 3.** *System (1) is FTS with respect to the parameters* $(c_1, c_2, T_f, R)$ *if and only if there exists a continuously differentiable piecewise function* $P : \mathbb{R} \to \mathbb{S}^n$ *such that for each* $t \in [0, T_f]$ *one has that*

$$\dot{P}(t) + A^T(t) P(t) + P(t) A(t) \prec 0, \tag{6}$$

$$P(t) \succeq R, \tag{7}$$

$$P(0) \prec \frac{c_2}{c_1} R. \tag{8}$$

*Remark* 4. It is interesting to observe that conditions (6) and (7) imply in the conditions

$$\dot{P}(t) + A^T(t) P(t) + P(t) A(t) \prec 0,\ P(t) \succ 0,\ \forall t \in \left[0, T_f\right]. \tag{9}$$

Thus, at first glance, it seems that if system (1) is FTS then it is asymptotically stable in the sense of Lyapunov. However, in general, this is not true. To see this, consider for instance, the scalar system

$$\dot{x}(t) = 2tx(t),\ x(t_0) = x_0 \tag{10}$$

with solution

$$x(t) = x_0\, e^{t^2 - t_0^2}.$$

Clearly, the equilibrium point in $x = 0$ is not asymptotically stable, thus there is no $P(t)$ satisfying (9) for all $t \geq 0$. Nevertheless, considering $P(t) = e^{-2t^2} - t/2$, one has that

$$\dot{P}(t) + A^T(t) P(t) + P(t) A(t) = -4te^{-2t^2} - \frac{1}{2} + 2(2t)\left(e^{-2t^2} - \frac{t}{2}\right) = -2t^2 - \frac{1}{2} < 0$$

is satisfied for all $t \geq 0$ and that besides $P(t) > 0$ not being satisfied for all $t \geq 0$, there exists a $T_f$ such that it is satisfied for all $t \in [0, T_f)$. Furthermore, for the above $P$, $P(0) = 1$ and it is always possible to choose $c_1 \geq 0$, $c_2 > 0$ and $R \succ 0$ such that (10) is FTS with respect to these parameters. ∎

In general, the conditions based on DMIs are not computationally tractable and a discretization similar to [26], in the context of the bounded real lemma, or to [15, 12], in the context of FTS/FTB, may be needed. According to [15, 12], the DMIs in Lemma 2 and Lemma 3 can be computationally implemented by restricting the variable $P(\cdot)$ as affine piecewise functions. With this constraint, the DMIs turn into several time dependent BMIs (or just LMIs in the case of merely FTS verification). The larger the number of different time subintervals for $P(\cdot)$, the larger the number of variables and BMI/LMI inequalities introduced in the conditions, possibly decreasing the conservatism. However, it is important to note that there is no numerical evidence or proof that if the number of different time subintervals tends to infinity, then the BMI/LMIs stemmed from the affine piecewise $P(\cdot)$ are equivalent to the original DMIs of Lemma 2 and Lemma 3. Furthermore, one should note that BMIs problems are difficult to be handled computationally due to its non-convex characteristic. It can be shown that even for small systems, the available BMI solvers may not find a solution (see Example 17).

For FTS, an alternative criterion less computationally expensive than the above DMIs and BMIs is given in [17]. This criterion is based on a differential Lyapunov equation (DLE) and stated in Lemma 5.

**Lemma 5.** *System (1) is FTS with respect to the parameters* $(c_1, c_2, T_f, R)$ *if and only if the continuously differentiable piecewise solution* $P : \mathbb{R} \to \mathbb{S}^n$ *of the DLE given by*

$$-\dot{P}(t) + A(t)P(t) + P(t)A^T(t) = 0,$$

$$P(0) = c_1 R^{-1}$$

*is positive definite and satisfies*

$$P(t) \prec c_2 R^{-1}$$

*for all* $t \in [0, T_f]$.

*Remark* 6. If the system (1) is autonomous and time-invariant, the solution of the DLE can be explicitly given by [27, p. 108]

$$P(t) = e^{At} P(0) e^{A^T t} = c_1 e^{At} R^{-1} e^{A^T t}.$$

Thus, system (1) is FTS with respect to the parameters $(c_1, c_2, T_f, R)$ if and only if the inequality

$$e^{At} R^{-1} e^{A^T t} \prec \frac{c_2}{c_1} R^{-1} \tag{11}$$

holds for each $t \in [0, T_f]$. Furthermore, if the function

$$\Gamma(t) = e^{At} R^{-1} e^{A^T t} - (c_2/c_1) R^{-1} \tag{12}$$

is convex, then (11) only needs to be checked at $t = T_f$. Summing up, if (1) is autonomous and time-invariant, a simple and sufficient criterion for a system to be FTS can be derived. This criterion uses only the parameters of the problem without any additional variable and it is given by

$$A^2 R^{-1} + R^{-1} (A^T)^2 + 2 A R^{-1} A^T \succeq 0,$$
$$e^{A T_f} R^{-1} e^{A^T T_f} \prec \frac{c_2}{c_1} R^{-1}.$$

■

Besides necessary and sufficient conditions for FTB and FTS, a considerable effort has been done in the literature to obtain computationally less expensive sufficient conditions based on LMIs. To the best of the authors knowledge, the FTB conditions of [9] for disturbance class $\mathscr{W}_d^\infty$ and the conditions of [7] for disturbance class $\mathscr{W}_d^2$ represent the least conservative LMI conditions in the literature for FTB analysis. The subsequent lemma encompasses both results.

**Lemma 7.** *Given a fixed $\beta > 0$, system (1) is FTB with respect to $(c_1, c_2, T_f, R, \mathscr{W}_d^\infty)$ (or with respect to $(c_1, c_2, T_f, R, \mathscr{W}_d^2)$) if there exist $Q_1 \in \mathbb{S}_+^n$, $Q_2 \in \mathbb{S}_+^r$, $\hat{\beta} > 0$ such that for each $t \in [0, T_f]$ one has that*

$$\begin{bmatrix} A(t)\tilde{Q}_1 + \tilde{Q}_1 A^T(t) - \beta\tilde{Q}_1 & B(t)Q_2 \\ \star & -\hat{\beta}Q_2 \end{bmatrix} \prec 0, \tag{13}$$

$$\frac{c_1}{\lambda_{min}(Q_1)} + \frac{d}{\lambda_{min}(Q_2)} < \frac{c_2 e^{-\beta T}}{\lambda_{max}(Q_1)}, \tag{14}$$

*where*

$$\tilde{Q}_1 := R^{-\frac{1}{2}} Q_1 R^{-\frac{1}{2}} \tag{15}$$

*and $\hat{\beta} = \beta$ (or $\hat{\beta} = 1$ respectively).*

For the particular case of class $\mathscr{W}_d^\infty$, Lemma 7 is a slight generalization of the original result of [9] where FTB was proved only for constant valued disturbances. The proof of Lemma 7 follows similar steps of Lemma 1 presented in [28].

The FTB characterization for the two classes of disturbance are closely related. For instance, when $\beta = 1$, both are equivalent. Another relation is given by the following proposition.

**Proposition 8.** *If a system is FTB with respect to $\mathscr{W}_d^\infty$ by the conditions of Lemma 7 with $0 < \hat{\beta} = \beta < 1$, then the system is also FTB with respect to $\mathscr{W}_d^2$. On the other hand, if the system is FTB with respect to $\mathscr{W}_d^2$ by the conditions of Lemma 7 with $\hat{\beta} = 1$ and $\beta > 1$, then the system is also FTB with respect to $\mathscr{W}_d^\infty$.*

*Proof.* Since the proof of the first and second statements are similar, only the proof of first one is presented. Let $\tilde{Q}_1$, $Q_1$, $Q_2$ and $0 < \hat{\beta} = \beta < 1$ be a particular solution of (13)-(15). Using Schur's complement arguments one has that (13) is equivalent to

$$A(t)\tilde{Q}_1 + \tilde{Q}_1 A^T(t) - \beta\tilde{Q}_1 \prec 0, \tag{16}$$

$$-\beta Q_2 - Q_2^T B^T(t)\left(A(t)\tilde{Q}_1 + \tilde{Q}_1 A^T(t) - \beta\tilde{Q}_1\right)^{-1} B(t) Q_2 \prec 0.$$

Since $0 < \beta < 1$, it follows that $\beta Q_2 \prec Q_2$ and

$$-Q_2^T B^T(t)\left(A(t)\tilde{Q}_1 + \tilde{Q}_1 A^T(t) - \beta\tilde{Q}_1\right)^{-1} B(t) Q_2 \prec Q_2. \tag{17}$$

Using Schur's complement arguments, from (16) and (17) it follows that $\tilde{Q}_1$, $Q_1$, $Q_2$ and $\beta$ are also a solution for (13) with $\hat{\beta} = 1$. □

An analysis for the finite time stability of a system can be obtained as a particular case of Lemma 7 by using $d = 0$ and $B = 0$. The conditions are explicitly stated in Corollary 9.

**Corollary 9.** *Given a fixed $\beta > 0$, system (1) is FTS with respect to $(c_1, c_2, T_f, R)$ if there exist $\hat{Q}_1 \in \mathbb{S}_+^n$ such that for each $t \in [0, T_f]$ one has that*

$$A(t)\tilde{Q}_1 + \tilde{Q}_1 A^T(t) - \beta\tilde{Q}_1 \prec 0, \tag{18}$$

$$\frac{\lambda_{max}\left(R^{\frac{1}{2}}\tilde{Q}_1 R^{\frac{1}{2}}\right)}{\lambda_{min}\left(R^{\frac{1}{2}}\tilde{Q}_1 R^{\frac{1}{2}}\right)} < \frac{c_2}{c_1} e^{-\beta T_f}. \tag{19}$$

*Remark* 10. Following [9], the inequalities (14) and (19) can be verified by appropriate LMIs. In fact, let $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$ be scalar variables. In the FTB case, the LMIs

$$\lambda_1 R \prec \tilde{Q}_1 \prec R,$$

$$\lambda_2 I \prec Q_2 \prec \lambda_3 I,$$
$$\begin{bmatrix} c_2 e^{-\beta T_f} & \sqrt{c_1} & \sqrt{d} \\ \star & \lambda_1 & 0 \\ \star & \star & \lambda_2 \end{bmatrix} \succ 0,$$

imply (14), and in the FTS case, the LMIs

$$\lambda_1 R \prec \tilde{Q}_1 \prec \lambda_2 R,$$

$$\lambda_2 < \frac{c_2}{c_1} e^{-\beta T_f} \lambda_1,$$

imply (19). ■

In view of the discussion so far, the analysis problem to be dealt with can be stated as follows.

**Problem 11.** Considering the trade-off between conservatism and numerical tractability, find FTB/FTS conditions less conservative than the ones from Lemma 7 but at same time less expensive computationally than the ones from Lemma 2.

In order to solve Problem 11, it is proposed to use the Finsler's lemma [25] to derive a new FTB/FTS analysis condition, which will be given in Theorem 13 of Section 3.

**Lemma 12** (Finsler's lemma)**.** *[25, 29] Let* $x \in \mathbb{R}^n$*,* $\mathscr{Q} \in \mathbb{S}^n$ *and* $\mathscr{B} \in \mathbb{R}^{m \times n}$ *such that rank* $(\mathscr{B}) = r < n$*. Let* $\mathscr{B}^\perp$ *a matrix whose columns span a basis for* $\mathrm{Ker}(\mathscr{B})$*, i.e. a matrix* $\mathscr{B}^\perp \in \mathbb{R}^{n \times (n-r)}$ *satisfying* $\mathrm{Im}\left(\mathscr{B}^\perp\right) = \mathrm{Ker}(\mathscr{B})$*. The following are equivalent:*

1. $x^T \mathscr{Q} x < 0 \; \forall x \in \mathbb{R}^n$ *such that* $x \neq 0$ *and* $\mathscr{B} x = 0$.
2. $\left(\mathscr{B}^\perp\right)^T \mathscr{Q} \mathscr{B}^\perp \prec 0$.

3. $\exists \mu \in \mathbb{R} : \mathscr{Q} - \mu \mathscr{B}^T \mathscr{B} \prec 0$.

4. $\exists \mathscr{X} \in \mathbb{R}^{n\times m} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^T \mathscr{X}^T \prec 0$.

The equivalence 1-4 of Lemma 12 allows the introduction of extra variables in the analysis condition, enlarging the search space and possibly leading to less conservative LMI conditions. In the context of Lyapunov asymptotic stability, this approach is originally done in [29]. To the best of the authors knowledge, it is the first time that this lemma is used in the context of finite time stability.

## 3. Main results

The following theorem presents LMI conditions for FTB that solves Problem 11.

**Theorem 13.** *Given a fixed $\beta > 0$, system (1) is FTB with respect to $\left(c_1, c_2, \mathscr{W}_d^\infty, T_f, R\right)$ (or with respect to $\left(c_1, c_2, \mathscr{W}_d^2, T_f, R\right)$) if there exist matrices $P_1 \in \mathbb{S}_+^n$, $P_2 \in \mathbb{S}_+^r$; matrices $F(t) \in \mathbb{R}^{n\times n}$, $G(t) \in \mathbb{R}^{n\times n}$, $H(t) \in \mathbb{R}^{r\times n}$; and a scalar $\hat{\beta} > 0$ such that*

$$\begin{bmatrix} \mathscr{L}_{11} & \tilde{P}_1 - F(t) + A^T(t) G^T(t) & F(t) B(t) + A^T(t) H^T(t) \\ \star & -G(t) - G^T(t) & G(t) B(t) - H^T(t) \\ \star & \star & -\hat{\beta} P_2 + H(t) B(t) + B^T(t) H^T(t) \end{bmatrix} \prec 0,$$
$$\mathscr{L}_{11} = -\beta \tilde{P}_1 + F(t) A(t) + A^T(t) F^T(t), \tag{20}$$

$$c_1 \lambda_{max}(P_1) + d\lambda_{max}(P_2) < c_2 e^{-\beta T_f} \lambda_{min}(P_1), \tag{21}$$

$$\tilde{P}_1 = R^{1/2} P_1 R^{1/2}, \tag{22}$$

*and $\hat{\beta} = \beta$ (or $\hat{\beta} = 1$, respectively).*

*Proof.* Consider the Lyapunov function candidate [7]

$$V(x) := x^T \tilde{Q}_1^{-1} x. \tag{23}$$

Proceeding similarly as in [7] one can show that if the inequalities

$$\dot{V}(x) < \beta V(x) + \hat{\beta} \omega^T Q_2^{-1} \omega, \tag{24}$$

$$\frac{c_1}{\lambda_{min}(Q_1)} + \frac{d}{\lambda_{min}(Q_2)} < \frac{c_2 e^{\beta T_f}}{\lambda_{max}(Q_1)} \tag{25}$$

are satisfied, then system (1) is FTB with respect to $\left(c_1, c_2, \mathscr{W}_d^{\infty}, T_f, R\right)$ (or with respect to $\left(c_1, c_2, \mathscr{W}_d^{2}, T_f, R\right)$), if $\hat{\beta} = \beta$ (or $\hat{\beta} = 1$, respectively). By defining $\tilde{P}_1 = \tilde{Q}_1^{-1}$ and $P_2 = Q_2^{-1}$, one has that (21) is equivalent to (25).

Now in order to use the Finsler's lemma, rewrite (24) as

$$\begin{bmatrix} x^T(t) & \dot{x}^T(t) & \omega^T(t) \end{bmatrix} \begin{bmatrix} -\beta \tilde{P}_1 & \tilde{P}_1 & 0 \\ \tilde{P}_1 & 0 & 0 \\ 0 & 0 & -\hat{\beta} P_2 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \omega(t) \end{bmatrix} < 0. \tag{26}$$

By using the equivalence 1-4 of Lemma 12 in (26) with

$$x \leftarrow \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \omega(t) \end{bmatrix}, \mathscr{Q} \leftarrow \begin{bmatrix} -\beta \tilde{P}_1 & \tilde{P}_1 & 0 \\ \tilde{P}_1 & 0 & 0 \\ 0 & 0 & -\hat{\beta} P_2 \end{bmatrix}, \mathscr{B}^T \leftarrow \begin{bmatrix} A^T \\ -I \\ B^T \end{bmatrix}, \mathscr{X} = \begin{bmatrix} F \\ G \\ H \end{bmatrix}$$

the inequality (20) follows immediately. □

When $d = 0$ and $B = 0$, one can use $H = 0$ in Theorem 13 to turn the FTB analysis condition into a FTS analysis condition. This is stated in the next corollary.

**Corollary 14.** *Given a fixed $\beta > 0$, system (1) is FTS with respect to $\left(c_1, c_2, T_f, R\right)$ if there exist matrix $P_1 \in \mathbb{S}_+^n$ and matrices $F(t) \in \mathbb{R}^{n\times n}$, $G(t) \in \mathbb{R}^{n\times n}$ such that*

$$\begin{bmatrix} -\beta\tilde{P}_1 + F(t)A(t) + A^T(t)F^T(t) & \tilde{P}_1 - F(t) + A^T(t)G^T(t) \\ \star & -G(t) - G^T(t) \end{bmatrix} \prec 0, \qquad (27)$$

$$\frac{\lambda_{max}\left(R^{-\frac{1}{2}}\tilde{P}_1 R^{-\frac{1}{2}}\right)}{\lambda_{min}\left(R^{-\frac{1}{2}}\tilde{P}_1 R^{-\frac{1}{2}}\right)} < \frac{c_2}{c_1} e^{-\beta T_f}. \qquad (28)$$

*Remark* 15. Similar to Remark 10, inequalities (21) and (28) can be guaranteed by appropriate LMIs. Let $\ell_1, \ell_2, \ell_3 > 0$ be scalar variables. In the FTB case, the LMIs

$$\ell_3 R \prec \tilde{P}_1 \prec \ell_1 R,$$

$$c_1\ell_1 + d\ell_2 < c_2 e^{-\beta T_f}\ell_3,$$

$$P_2 \prec \ell_2 I,$$

imply (21), and in the particular case of FTS,

$$\ell_3 R \prec \tilde{P}_1 \prec \ell_1 R,$$

$$\ell_1 < \frac{c_2}{c_1} e^{-\beta T_f}\ell_3,$$

imply (28). ■

In the next proposition, it is shown that the FTB characterization of Theorem 13 are in the worst case as stringent as the characterization of Lemma 7 which, to the best of the authors knowledge are the current less conservative LMI condition for FTB in the literature. However, it is important to note that the computational properties are different. In fact, in Example 17 it is shown a system for which the LMIs of Theorem 13 are feasible but the LMIs of Lemma 7 are not.

**Proposition 16.** *For any fixed $\varepsilon > 0$, if $P_1$, $P_2$, $F = \tilde{P}_1$, $G = \varepsilon I$ and $H = 0$ satisfy the conditions of Theorem 13, then $Q_1 = P_1^{-1}$ and $Q_2 = P_2^{-1}$ satisfy the conditions of Lemma 7.*

*Proof.* Let $\varepsilon > 0$. Choosing $F = \tilde{P}_1$, $G = \varepsilon I$ and $H = 0$ in Theorem 13 we have that LMI (20) becomes

$$\begin{bmatrix} -\beta \tilde{P}_1 + \tilde{P}_1 A(t) + A^T(t) \tilde{P}_1 & \varepsilon A^T(t) & \tilde{P}_1 B(t) \\ \star & -2\varepsilon I & \varepsilon B(t) \\ \star & \star & -\hat{\beta} P_2 \end{bmatrix} \prec 0. \tag{29}$$

In view of this, there is a sufficiently small $\varepsilon$ such that

$$\begin{bmatrix} -\beta \tilde{P}_1 + \tilde{P}_1 A(t) + A^T(t) \tilde{P}_1 & \tilde{P}_1 B(t) \\ \star & -\hat{\beta} P_2 \end{bmatrix} \prec 0, \tag{30}$$

which is equivalent to LMI (13) from Lemma 7.

Multiplying (29) from the left and right by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix},$$

one has that

$$\begin{bmatrix} -\beta \tilde{P}_1 + \tilde{P}_1 A(t) + A^T(t) \tilde{P}_1 & \tilde{P}_1 B(t) & \varepsilon A^T(t) \\ \star & -\beta P_2 & \varepsilon B^T(t) \\ \star & \star & -2\varepsilon I \end{bmatrix} \prec 0. \tag{31}$$

By Schur's complement arguments, (31) is equivalent to

$$\begin{bmatrix} -\beta \tilde{P}_1 + \tilde{P}_1 A(t) + A^T(t) \tilde{P}_1 & \tilde{P}_1 B(t) \\ \star & -\beta P_2 \end{bmatrix} \prec -\frac{1}{2}\varepsilon \begin{bmatrix} A^T(t) \\ B^T(t) \end{bmatrix} \begin{bmatrix} A(t) & B(t) \end{bmatrix} \preceq 0,$$

from which follows the result.

Finally, replacing $\tilde{P}_1 = \tilde{Q}_1^{-1}$ and $P_2 = Q_2^{-1}$ in (30) and multiplying to the left and to the right by

$$\begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & Q_2 \end{bmatrix},$$

one has that (30) is equivalent to (13). □

Finally, the computational complexity of the LMI-based and BMI-based algorithms, in terms of scalar variables and rows, are summarized in Table 1. The unique features of each method for FTB analysis are highlighted in Table 2.

| **Conditions for FTB**[1] | **Number of rows** | **Number of scalar variables** |
|---|---|---|
| Lemma 2 | $(4M+1)n$ | —[2] |
| Lemma 7 | $3n+3r+1$ | $\frac{1}{2}(n^2+r^2+n+r)+3$ |
| Theorem 13 | $4n+3r+1$ | $\frac{5}{2}n^2+nr+\frac{1}{2}r^2+\frac{1}{2}(n+r)+3$ |
| **Conditions for FTS** | | |
| Corollary 3 | 5Mn | $Mn(n+1)$ |
| Corollary 9 | 3n+2 | $\frac{n(n+1)}{2}+2$ |
| Corollary 14 | 4n+2 | $\frac{n(n+1)}{2}+2n^2+3$ |

Table 1: Computational complexity of each condition in terms of number of rows and scalar variables. The number $M$ denotes the level of discretization of the DMI of Lemma 2.

[1]The number of rows and scalar variables are counted after applying Remark 10 (respectively Remark 15) to relax Lemma 7 and Corollary 9 (respectively Theorem 13 and Corollary 14) to LMI conditions.

[2]Since the discretization of Lemma 2 yields in general a BMI problem, the number of scalar variables is not directly comparable to the number of scalar variables of a LMI problem.

| | Condition type | Advantages and disadvantages |
|---|---|---|
| Lemma 2 | BMI | The conservatism may be decreased with the increase of the level of discretization. |
| | | BMI is a non-convex optimization problem. |
| Lemma 7 | LMI | Use less variables and rows than Theorem 13. |
| | | It may be more conservative than Theorem 13. |
| Theorem 13 | LMI | Use more variables and rows than Lemma 7. |
| | | It may be less conservative than Lemma 7. |

Table 2: Unique features of each approach for FTB analysis.

## 4. Numerical examples

In this section some examples comparing the current best FTB/FTS conditions and the proposed conditions are presented.

In the first example, the FTB conditions are compared.

**Example 17.** Consider the system (1) with matrices given by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and with finite time parameters given by $c_1 = 1$, $c_2 = 6$, $\mathscr{W} = \mathscr{W}_d^\infty$ with $d = 1$, $R = I$, $\beta = 22$ and $T_f = 0.1$s. It is important to mention that this system is asymptotically unstable in infinite time horizon.

By using the parser YALMIP[3] [30] within Matlab environment, it is possible to easily implement the LMI conditions of Lemma 7 or Theorem 13 in the solver

[3] Available to download at `http://users.isy.liu.se/johanl/yalmip/`.

| FTB conditions | Number of rows | Number of scalar variables |
|---|---|---|
| Lemma 2 | $12M+3$ | — |
| Lemma 7 | 13 | 10 |
| Theorem 13 | 16 | 31 |

Table 3: Computational burden of each FTB condition in Example 17.

SeDuMi[4] [31], or the DMI conditions of Lemma 2 (after replacing $P(\cdot)$ by affine piecewise functions) in the BMI solver PENLAB[5] [32].

On one hand, the LMI-based conditions uses 10 scalar variables and 13 LMI rows for Lemma 7 and 31 scalar variables and 16 LMI rows for Theorem 13. Although the LMI of Lemma 7 is infeasible, the LMI of Theorem 13 assures that the system is FTB with respect to the given finite time parameters.

On the other hand, the DMI-based Lemma 2 needs $Mn(n+1)$ scalar variables and $(4M+1)n$ rows, where $M$ is the number of different patches used for the variable $P(\cdot)$. Even using a big number for $M$ , the solver PENLAB could not find a feasible function to the BMI feasibility problem and consequently, it was not possible to guarantee that the system is FTB. The computational burden of each method is summarized in Table 3.

Further, in order to graphically illustrate the finite time boundedness, a time-simulation was performed considering 100 random initial conditions whose norm is less than or equal to $c_1 = 1$. Considering the set of all trajectories $x(t)$ stemmed from these initial conditions, the maximum $x^T(t)x(t)$ at each $t$ is shown in Figure 1. As can be seen, every trajectory maintains its norm below $c_2 = 6$ during the

[4]Available to download at `http://sedumi.ie.lehigh.edu/`.

[5]Available to download at `http://web.mat.bham.ac.uk/kocvara/penlab/`.

prescribed time interval of $t \in [0, 0.1]$.

In the next example, FTS conditions are compared.

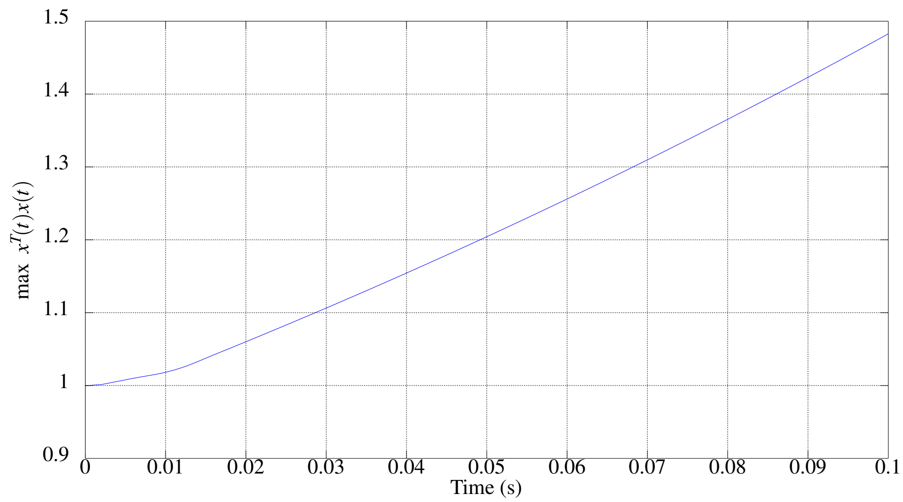


Figure 1: Time simulation of system of Example 17.

**Example 18.** Consider the system (1) with

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$B = 0$ and $R = I$. Since this case deals with only the finite time stability (and not finite time boundedness) of an autonomous system, it is possible to use Lemma 5 and Remark 6. Note, however, that the function (12) for this choice of $A$ and $R$ is not convex, thus the verification of $\Gamma(t)$ only at $t = T_f$ is not sufficient. In fact, one can check that $\Gamma(t)$ is not negative definite for $2.57 \leq t < 3.72$, besides being negative definite for $3.72 \leq t \leq 8$. Thus, the testing of the finite time stability of the system for $T = 8$ checking the negative definiteness of $\Gamma(t)$ only at points $t \notin [2.57, 3, 72]$ leads to a wrong conclusion about the finite time stability of the system.

Even so, since the implementation of Corollaries 3, 9 and 14 using LMIs leads to only sufficient conditions, those also can not be used to conclude that the system is not FTS and the only way to assure this conclusion is by sufficiently discretizing the interval $[0, T_f]$ when using Lemma 5.

In the next, FTB conditions are applied in a practical system.

**Example 19.** Following Section 2.5 of [5], we apply the proposed technique to analyze the finite time boundedness of a vehicle active suspension system in order to prevent excessive suspension bottoming. The model of the suspension system, illustrated in Figure 2, is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{K_s}{M_s} & -\frac{B_s}{M_s} & 0 & \frac{B_s}{M_s} \\ 0 & 0 & 0 & 1 \\ \frac{K_s}{M_u} & \frac{B_s}{M_u} & -\frac{K_u}{M_u} & -\frac{B_s}{M_u} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \omega,$$

where the state variables are given by $x_1 = x_s - x_u$ (the suspension stroke), $x_2 = \dot{x}_s$, $x_3 = x_u - x_o$ (the tire deflection) and $x_4 = \dot{x}_u$. The parameters $M_s$ and $M_u$ are respectively the sprung and the unsprung mass, $K_u$ is the tire stiffness and $(K_s, B_s)$ consist of the passive suspension [33]. The values of the model parameters are [5]

$$M_s = 320\,\text{kg},\ K_s = 50\,\frac{\text{kN}}{\text{m}},\ B_s = 2000\,\frac{\text{N s}}{\text{m}},$$
$$K_u = 200\,\frac{\text{kN}}{\text{m}},\ M_u = 40\,\text{kg},\ u_{max} = 100\,\text{kN}$$

and the FTB parameters are $c_1 = 1$, $c_2 = 6$, $d = 1$, $T_f = 0.2$ and $R = I$.

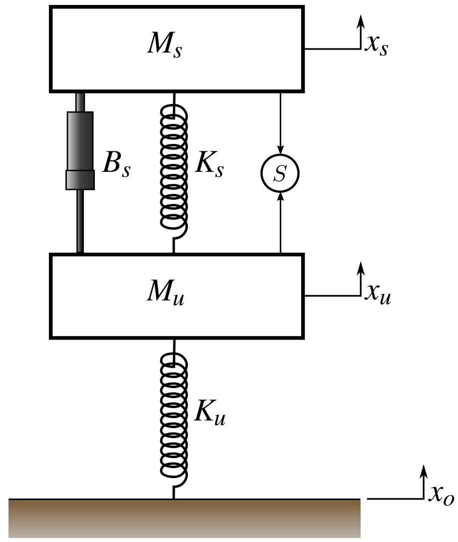


Figure 2: Vehicle active suspension system.

Although it was not possible to guarantee the finite time boundedness of the car suspension system using Lemma 7 or Lemma 2, the use of Theorem 13 with $\beta = 0.3$ yields the conclusion that the system is FTB.

## 5. Conclusion

In this paper, the problem of finite time boundedness was investigated. A review of the FTB and FTS analysis conditions using DMI, BMI, DLE or LMI was done, presenting some propositions and examples that helps to compare those analysis conditions. By using the Finsler's lemma, it was possible to derive a new condition based on LMIs for a system to be FTB. This condition was shown to be less conservative than other LMI conditions in the literature. Further works should investigate the synthesis of FTS and FTB controllers and filters using the proposed conditions. Additionally, possible extensions of the current results for systems with time-delays, as done in [34], might also be investigated.

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