Parameter-Dependent Filter with Finite Time Boundedness Property for Continuous-Time LPV Systems ★

Hugo T. M. Kussaba * **Renato A. Borges** * **João Y. Ishihara** *,**

* *Department of Electrical Engineering, University of Brasília – UnB, 70910-900, Brasília, DF, Brazil (e-mail: kussaba@lara.unb.br,{raborges,ishihara}@ene.unb.br).*
** *Department of Electrical Engineering, University of California, Los Angeles (UCLA) – 90095, California, LA, United States of America*

AbstractIn this paper the problem of stable estimation for linear parameter varying (LPV) systems in finite-time setting is considered. In order to bound the states and the estimation errors, a parameter-dependent filter is proposed. Under the assumption that the parameters variation are sufficiently small, the filter existence and synthesis is characterized by linear matrix inequalities (LMI) conditions. A numerical example is provided to illustrate the proposed technique.

Keywords: Filter design, Disturbance signals, Uncertain linear systems, Linear parameter varying systems, Finite time stability, Finite time boundedness, Linear matrix inequality.

1. INTRODUCTION

Considering the challenges posed by the constant technological development, there is no doubt that the improvement of filtering techniques is of considerable importance. The use of dynamic filters to remove unwanted signal characteristics or to estimate system information from corrupted measurements is increasing within engineering. In many practical applications, this means suppressing interfering signals and reducing the effect of external noise in communication systems, electronic devices, industrial plants, among others. In fact, filter performance has been investigated under several scenarios like nonlinearities, delays, and parameter uncertainties. However, one can note that most works in the literature consider system performance only after the system had run a large amount of time, that is, filters and controllers are designed to achieve their goal only asymptotically. This is a strong theoretical limitation since for many practical applications it is important that the overall system achieves a desired state in a specified finite time. In order to fill this gap, many infinite time concepts like stability and controllability have been extended for finite time setting. In particular, a similar notion of ultimately bounded system Khalil (2002) in the finite time setting is the notion of Finite Time Stability, and in the presence of disturbances, the notion of Finite Time Boundedness (Amato et al., 2001).

The notion of Finite Time Boundedness states that the time-varying linear system

$$\dot{x}(t) = A(t)x(t) + G(t)w(t),\ \forall t \in [0,\ T] \tag{1}$$

subject to a disturbance w in a pre-specified class $\mathscr{W}$ is Finite Time Bounded (also abbreviated as FTB) with respect to $(c_1, c_2, T, R, \mathscr{W})$, with $c_2 > c_1$ and $R > 0$, if

$$x'(0)Rx(0) \leq c_1 \Rightarrow x'(t)Rx(t) \leq c_2,\ \forall t \in [0,\ T], \forall w \in \mathscr{W}.$$

In the particular case that the pre-specified class $\mathscr{W}$ is empty or $G = \mathbf{0}$, the system is said to be Finite Time Stable (FTS). Still in Amato et al. (2001), sufficient conditions for (1) being FTB are also derived in the form of a linear matrix inequality (LMI) feasibility problem. Those conditions are used for the synthesis of a state feedback controller which assures that the closed loop system is FTB. Further works in the area propose variations of the FTB definition, or of the structures of the controllers and the plant, or of the classes $\mathscr{W}$ of disturbances.

In this paper is considered a more general situation which the designer has to face an uncertain ambient—besides disturbances, the system itself has its parameters not known exactly. More specifically, it is supposed that the system is a continuous-time linear parameter varying (LPV) system. This setting is interesting because many general nonlinear systems can be converted into a LPV form (Toth, 2010). The examples range from flight and automative systems (Ganguli et al., 2002; Baslamisli et al., 2009) to anesthesia delivery (Lin et al., 2008) and diabetes control (Peña and Ghersin, 2010).

Considering the filter design problem, the goal is to guarantee that the estimation error is FTB with respect to $(c_1, c_2, T, R, \mathscr{W})$, as done in Luan et al. (2010) for stochastic systems; He and Liu (2011) for time-delay jump systems and Liu et al. (2012) for singular stochastic systems. In all these cases, the filters are designed using LMI conditions to ensure the FTB property. In principle, a very strong necessary and sufficient FTB filter design conditions could be developed based on the differential linear matrix inequality (DLMI) characterization (Amato et al., 2003, 2005, 2014) for FTB. However, this development has two drawbacks. First, it is important to salient that the DLMI approach for filtering is not trivial since the filter structure demands an analysis with an input signal, a much more challenging case than that considered by Amato et al. (2003). And second, it is noted that DLMI problems are generally computationally very expensive and in most situations even prohibitive.

★ Supported by the Brazilian agencies CAPES and CNPq

Therefore, it is interesting the search for solutions that uses less computational effort than the DLMIs. Thus, in this paper we aim achieving this goal by using only the LMI framework.

Under the assumption that the parameters of the plant are sufficiently slow time-varying, a new synthesis condition for a homogeneous polynomially parameter-dependent FTB filter for continuous-time LPV systems is derived in this paper. Those systems are an indexed collection of linear systems in which the indexing parameter is independent of the state (Shamma, 2012). Depending on the scenario, this indexing parameter can be seen as a parametric uncertainty of the model or a measurable parameter possibly read in real time, which can be used in the design of a controller or a filter that accounts for all possible variations of this parameter. Furthermore, the proposed approach also considers that for a particular case where the LMIs depend on a parameter α in the unit simplex, homogeneous polynomial structures can be used in the search for less conservative sets of design conditions, as done in Oliveira and Peres (2007).

This paper is organized as follows. Detailing of the problem and auxiliary lemmas are presented in Section 2. The main theorem, where LMI conditions are derived for the synthesis of a filter that solves the FTB problem for LPV systems, is proved in Section 3. A numerical example is given in Section 4 to illustrate the application of the technique. Finally, the conclusion is presented in Section 5.

In the sequel the following notation will be used: The symbol $(')$ indicates the transpose of a matrix; $P > 0$ means that P is symmetric positive definite. $\mathbb{R}$ represents the set of real numbers, $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ the set of nonnegative integers. $\mathrm{card}(\cdot)$ denotes the cardinality of a set. $\lambda_{max}(\cdot)$ and $\lambda_{min}(\cdot)$ indicate, respectively, the maximum and the minimum eigenvalue of the argument. The term $(\star)$ indicates symmetric terms in the LMIs and $\mathbf{I}$ and $\mathbf{0}$ are the identity and the zero matrices of suitable dimensions.

2. PROBLEM STATEMENT AND PRELIMINARY RESULTS

Consider a LPV system with $t \in [0,\ T]$ and

$$\begin{aligned} \dot{x}(t) &= A(\alpha(t))x(t) + B(\alpha(t))w(t), \\ y(t) &= C_y(\alpha(t))x(t) + D(\alpha(t))w(t), \\ z(t) &= C_z(\alpha(t))x(t), \end{aligned} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state space vector, $y(t) \in \mathbb{R}^q$ is the measured output, $z(t) \in \mathbb{R}^p$ is the signal to be estimated and $w(t) \in \mathbb{R}^r$ is the noise input with bounded L_2 norm. The parameter $\alpha(t)$ is assumed to be available online and is continuous with respect to its time dependence—which, to lighten the notation, will be omitted wherever there is no ambiguity.

All matrices are real, with appropriate dimensions and belongs to the polytope $\mathscr{P}$

$$\left[\begin{array}{c|c} A(\alpha) & B(\alpha) \\ \hline C_y(\alpha) & D(\alpha) \\ \hline C_z(\alpha) & - \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c} A_i & B_i \\ \hline C_{yi} & D_i \\ \hline C_{zi} & - \end{array}\right]. \tag{3}$$

For all $t \in [0,\ T]$, the system matrices are given by the convex combination of the known vertices of the polytope $\mathscr{P}$.

The vector of time-varying parameters $\alpha \in \mathbb{R}^N$ belongs for all $t \in [0,\ T]$ to the unit N-simplex Δ_N, that is:

$$\Delta_N = \left\{ \theta \in \mathbb{R}^N : \sum_{i=0}^{N} \theta_i = 1,\ \theta_i \geq 0 \right\}.$$

To account the information given by the parameter α, parameter-dependent matrices are used in the dynamics of a full order proper filter, precisely:

$$\begin{aligned} \dot{x}_f(t) &= A_f(\alpha)x_f(t) + B_f(\alpha)y(t), \\ z_f(t) &= C_f(\alpha)x_f(t), \end{aligned} \tag{4}$$

where $x_f(t) \in \mathbb{R}^n$ is the filter state and $z_f(t) \in \mathbb{R}^p$ the estimated signal. Coupling the filter to the plant, the equations that describe the augmented system dynamics are given by

$$\begin{aligned} \dot{\varsigma}(t) &= \bar{A}(\alpha)\varsigma(t) + \bar{B}(\alpha)w(t), \\ e(t) &= \bar{C}(\alpha)\varsigma(t), \end{aligned} \tag{5}$$

where $\varsigma(t) = [x(t)'\ \ x_f(t)']'$, $e(t) = z(t) - z_f(t)$, and

$$\bar{A}(\alpha) = \begin{bmatrix} A(\alpha) & \mathbf{0} \\ B_f(\alpha)C_y(\alpha) & A_f(\alpha) \end{bmatrix},\ \bar{B}(\alpha) = \begin{bmatrix} B(\alpha) \\ B_f(\alpha)D(\alpha) \end{bmatrix},$$
$$\bar{C}(\alpha) = [C_z(\alpha)\ \ -C_f(\alpha)].$$

It is desirable that both the state of the plant and the error between it and the state of the filter are bounded during a finite time horizon. This motivates Definition 1.

Definition 1. Given three positive scalars c_1, c_2 and T, with $c_2 > c_1$, positive definite matrices $R_p \in \mathbb{R}^{n\times n}$ and $R_e \in \mathbb{R}^{n\times n}$ and the class of signals $\mathscr{W}_d$, the LPV system

$$\dot{\varsigma}(t) = \bar{A}(\alpha)\varsigma(t) + \bar{B}(\alpha)w(t) \tag{6}$$

is FTB with respect to $(c_1, c_2, \mathscr{W}_d, T, R_p, R_e)$, if

$$\begin{bmatrix} x(0) \\ x(0) - x_f(0) \end{bmatrix}' \begin{bmatrix} R_p & \mathbf{0} \\ \mathbf{0} & R_e \end{bmatrix} \begin{bmatrix} x(0) \\ x(0) - x_f(0) \end{bmatrix} \leq c_1$$

implies that

$$\begin{bmatrix} x(t) \\ x(t) - x_f(t) \end{bmatrix}' \begin{bmatrix} R_p & \mathbf{0} \\ \mathbf{0} & R_e \end{bmatrix} \begin{bmatrix} x(t) \\ x(t) - x_f(t) \end{bmatrix} \leq c_2$$

for all $w \in \mathscr{W}_d$ and for all $t \in [0,\ T]$.

Remark 1. It should be noted that the definition of a FTB system presented here is a specialization for LPV systems of the definition presented in Amato et al. (2001) with R chosen as:

$$R = \begin{bmatrix} R_p + R_e & -R_e \\ -R_e & R_e \end{bmatrix}. \tag{7}$$

The matrices R_p and R_e can be seen as weighting matrices that set the importance between bounding the states of the plant and the error between it and the states of the filter. In contrast to a classical scenario in which the Lyapunov stability of $x - x_f$ implies the Lyapunov stability of $z - z_f$ using a Luenberger observer as a filter, constraining the error $x - x_f$ in a region during a finite time does not necessarily imply that $z - z_f$ satisfies the same constraining. In fact, $z - z_f$ can be bounded independently of $x - x_f$, motivating the next definition.

Definition 2. Given a symmetric positive definite matrix Ω, the filter (4) is said to be Ω-bounded in finite time T if its estimation error $e(t)$ satisfies

$$e'(t)e(t) < \varsigma'(t)\Omega^{-1}\varsigma(t),\ \ \forall t \in [0,\ T]. \tag{8}$$

Taking into account the above discussion, the FTB filtering problem to be solved in this work is formally stated as follows.

Problem 1. Assuming that $\alpha \in \Delta_N$ is available online for every $t \in [0,\ T]$ and its variation is sufficiently small, find matrices $A_f(\alpha)$, $B_f(\alpha)$ and $C_f(\alpha)$ in (4), such that the augmented system (5) is FTB with respect to $(c_1, c_2, \mathscr{W}_d, T, R_p, R_e)$ and the estimation error is Ω-bounded in finite time T.

The subsequent lemma, from Amato et al. (2001) with an extended proof in Borges et al. (2013) considering a wider class of noises, presents a sufficient condition to analyze if a LPV system is FTB and it is used in the solution of Problem 1.

Lemma 1. For a sufficiently slow varying parameter α, system (6) is FTB with respect to $(c_1, c_2, \mathscr{W}_d, T, R_p, R_e)$, if, for all $\alpha \in \Delta_N$, there exist positive definite symmetric matrices $Q_1(\alpha) \in \mathbb{R}^{2n\times 2n}$, $Q_2(\alpha) \in \mathbb{R}^{r\times r}$ and a positive scalar β such that

$$\begin{bmatrix} \bar{A}(\alpha)\tilde{Q}_1(\alpha) + \tilde{Q}_1(\alpha)\bar{A}'(\alpha) - \beta\tilde{Q}_1(\alpha) & \bar{B}(\alpha)Q_2(\alpha) \\ (\star) & -\beta Q_2(\alpha) \end{bmatrix} < \mathbf{0}, \tag{9}$$

$$\frac{c_1}{\lambda_{min}[Q_1(\alpha)]} + \frac{d}{\lambda_{min}[Q_2(\alpha)]} < \frac{c_2 e^{-\beta T}}{\lambda_{max}[Q_1(\alpha)]}, \tag{10}$$

in which $\tilde{Q}_1(\alpha) = R^{-1/2}Q_1(\alpha)R^{-1/2}$, with R given by (7).

3. MAIN RESULTS

The following theorem presents sufficient conditions, in terms of a parameter-dependent LMI, for the synthesis of matrices that solves Problem 1.

Theorem 1. Given a LPV continuous-time system (2), parameters $(c_1, c_2, d, T, R_p, R_e)$ and a fixed scalar parameter β, if, for each $\alpha \in \Delta_N$, there exist symmetric positive definite matrices $K \in \mathbb{R}^{n\times n}$, $W(\alpha) \in \mathbb{R}^{r\times r}$ and $Z(\alpha) \in \mathbb{R}^{n\times n}$; matrices $L(\alpha) \in \mathbb{R}^{n\times q}$, $M(\alpha) \in \mathbb{R}^{n\times n}$ and $F(\alpha) \in \mathbb{R}^{p\times n}$ and positive real scalars μ_1, μ_2 and μ_3, such that

$$\begin{bmatrix} \mathscr{M}_{11}(\alpha) & \mathscr{M}_{12}(\alpha) & KB(\alpha) + L(\alpha)D(\alpha) \\ (\star) & \mathscr{M}_{22}(\alpha) & Z(\alpha)B(\alpha) \\ (\star) & (\star) & -\beta W(\alpha) \end{bmatrix} < \mathbf{0},$$
$$\mathscr{M}_{11}(\alpha) = -\beta K - M(\alpha) - M'(\alpha),$$
$$\mathscr{M}_{12}(\alpha) = KA(\alpha) + L(\alpha)C_y(\alpha) + M(\alpha),$$
$$A'(\alpha)Z(\alpha) + Z(\alpha)A(\alpha) - \beta Z(\alpha), \tag{11a}$$

$$c_1\mu_1 + d\mu_3 < c_2 e^{-\beta T}\mu_2, \tag{11b}$$
$$W(\alpha) < \mu_3\mathbf{I}, \tag{11c}$$
$$\mu_2 R_p < Z(\alpha) < \mu_1 R_p, \tag{11d}$$
$$\mu_2 R_e < K < \mu_1 R_e, \tag{11e}$$
$$\begin{bmatrix} K & \mathbf{0} & F'(\alpha) \\ (\star) & Z(\alpha) & C_z'(\alpha) - F'(\alpha) \\ (\star) & (\star) & \mathbf{I} \end{bmatrix} > \mathbf{0}, \tag{11f}$$

then for a sufficiently slow varying parameter α there exists a filter in the form (4), such that the augmented system (5) is FTB with respect to $(c_1, c_2, \mathscr{W}_d, T, R_p, R_e)$ and the filter is also Ω-bounded in finite time T for $\Omega = \Gamma\tilde{Q}_1\Gamma'$, $\Gamma = diag(\mathbf{I}, \Gamma_{22})$, with Γ_{22} non-singular. A realization of the filter is given by the matrices:

$$A_f(\alpha) = -K^{-1}M(\alpha),$$
$$B_f(\alpha) = -K^{-1}L(\alpha), \tag{12}$$
$$C_f(\alpha) = F(\alpha)\Gamma_{22}^{-1}.$$

Proof. As presented in Chilali and Gahinet (1996) in the context of pole placement, consider the partitioned matrices

$$\tilde{Q}_1(\alpha) = \begin{bmatrix} X(\alpha) & U'(\alpha) \\ U(\alpha) & \hat{X}(\alpha) \end{bmatrix}, \tilde{Q}_1^{-1}(\alpha) = \begin{bmatrix} Y(\alpha) & V'(\alpha) \\ V(\alpha) & \hat{Y}(\alpha) \end{bmatrix},$$
$$H(\alpha) = \begin{bmatrix} Y(\alpha) & \mathbf{I} \\ V(\alpha) & \mathbf{0} \end{bmatrix},$$

together with the following change of variables

$$M(\alpha) = -KA_f(\alpha)U(\alpha)Z(\alpha), \tag{13a}$$
$$L(\alpha) = -KB_f(\alpha), \tag{13b}$$
$$F(\alpha) = C_f(\alpha)\Gamma_{22}U(\alpha)Z(\alpha), \tag{13c}$$
$$W(\alpha) = Q_2^{-1}(\alpha), \tag{13d}$$

where $X(\alpha)$, $Y(\alpha)$ and $Q_2^{-1}(\alpha)$ are chosen such that

$$Z(\alpha) = X^{-1}(\alpha),$$
$$W(\alpha) = Q_2^{-1}(\alpha),$$
$$K = Y(\alpha) - Z(\alpha).$$

By multiplying the LMI (11a) on the left by $\bar{H}'(\alpha)$ and on the right by $\bar{H}(\alpha)$, and multiplying the result on the left by $\tilde{H}'(\alpha)$ and on the right by $\tilde{H}(\alpha)$, with

$$\bar{H}(\alpha) = \begin{bmatrix} N(\alpha) & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix}, \tilde{H}(\alpha) = \begin{bmatrix} H^{-1}(\alpha) & \mathbf{0} \\ \star & \mathbf{I} \end{bmatrix},$$
$$N(\alpha) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & X(\alpha) \end{bmatrix},$$

the LMI (9) is obtained. Moreover, it is easy to see that LMI (10) is satisfied if the conditions

$$c_1\mu_1 + d\mu_3 < c_2 e^{-\beta T}\mu_2, \tag{14}$$
$$Q_2^{-1}(\alpha) < \mu_3\mathbf{I}, \tag{15}$$
$$\mu_2\mathbf{I} < Q_1^{-1}(\alpha) < \mu_1\mathbf{I}, \tag{16}$$

are guaranteed.

Inequalities (14) and (15) are LMIs (11b) and (11c), respectively. By multiplying inequality (16) on the left and on the right by $R^{1/2}$, with R given by (7), one has

$$\mu_2 R < \tilde{Q}_1^{-1}(\alpha) < \mu_1 R. \tag{17}$$

Knowing that the identity $\tilde{Q}_1(\alpha)\tilde{Q}_1^{-1}(\alpha) = \mathbf{I}$ gives the equations

$$X(\alpha)Y(\alpha) + U'(\alpha)V(\alpha) = \mathbf{I},$$
$$X(\alpha)V'(\alpha) + U'(\alpha)\hat{Y}(\alpha) = \mathbf{0},$$

one has for $U(\alpha) = X(\alpha)$ that

$$V(\alpha) = -K,$$
$$\hat{Y}(\alpha) = K.$$

Consequently, inequality (17) is satisfied if, and only if

$$\mu_2 R < \begin{bmatrix} K + Z(\alpha) & -K \\ -K & K \end{bmatrix} < \mu_1 R. \tag{18}$$

Left-multiplying LMI (18) by G' and right-multiplying by G, with

$$G = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \end{bmatrix}$$

one can see that inequality (18) is equivalent to LMIs (11d) and (11e). At last, by multiplying LMI (11f) on the left by $\bar{H}'(\alpha)$ and on the right by $\bar{H}(\alpha)$, multiplying the result on the left by $\tilde{J}'(\alpha)$ and on the right by $\tilde{J}(\alpha)$, with

$$\tilde{J}(\alpha) = \begin{bmatrix} J^{-1}(\alpha)\Gamma_{22}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, J(\alpha) = \begin{bmatrix} \mathbf{I} & X(\alpha) \\ \mathbf{0} & X(\alpha) \end{bmatrix}$$

and then applying Schur complement in the resulting matrix, one has

$$e'(t)e(t) < \varsigma'(t)\Omega^{-1}\varsigma(t),$$

which guarantees the constraint in the estimation error.

By the choice of $U(\alpha)$, one has that $U(\alpha)Z(\alpha) = \mathbf{I}$ and that the filter matrices $A_f(\alpha)$, $B_f(\alpha)$ and $C_f(\alpha)$ can be recovered from the change of variables in (13).

Remark 2. It should be remarked that if β is not fixed, the proposed conditions are not longer linear for a fixed α. Actually, they are not even bilinear due to (11b). Aside from that, a binary search for a suitable β can be guided by balancing (11a) and (11b) and should not be a computational burden since β is just a scalar variable.

In the definition of Problem 1, Ω is a design parameter that must be adjusted for a suitable weighting between the size of output estimation error and the size of the filter and plant states. It should be observed that there is no loss of generality to write $\Omega = \Gamma\tilde{Q}_1\Gamma'$ and to consider the parameter Γ as invertible (Ω, Γ, $\tilde{Q}_1$ are invertible). The choice of $\Gamma = \mathrm{diag}(\mathbf{I}, \Gamma_{22})$ allows to directly adjust the matrix C_f with a scale factor given by Γ_{22}. Consequently, the quality of the realization $\{A_f, B_f, C_f\}$ of the filter can be improved without deteriorating the estimate error.

Theorem 1 leads to a LMI feasibility problem that must be satisfied for all parameters $\alpha \in \Delta_N$. Although this is an infinite dimension problem in the parameter α, the fact that it lies in the unit N-simplex can be used to find sufficient LMI conditions written only in terms of the vertices of the polytope (Bliman et al., 2006).

In fact, using the relaxation proposed in Oliveira and Peres (2007) one can write the parameter-dependent LMIs in Theorem 1 as LMIs that are independent of the parameter α. As the level of relaxation increases, it is possible to achieve less conservative sets of conditions and tending to necessary and sufficient conditions.

For this purpose, the next Definition 3 generalizes the linear dependence on the parameter α to a homogeneous polynomial dependence.

Definition 3. A matrix $M(\alpha)$ is homogeneous polynomially parameter-dependent (HPPD) on $\alpha \in \Delta_N$ with degree g if it can be expressed as

$$M(\alpha) = \sum_{k \in \mathscr{S}_g} \alpha_1^{k_1} \alpha_2^{k_2} \cdots \alpha_N^{k_N} M_k, \tag{19}$$

with

$$\mathscr{S}_g = \left\{ k \in (\mathbb{Z}_+)^N : \sum_{i=1}^{N} k_i = g \right\}.$$

M_k are the matrices coefficients of the monomials of $M(\alpha)$, where

$$\mathrm{card}(\mathscr{S}_g) = \frac{(N+g-1)!}{g!\,(N-1)!}.$$

The set of HPPD of $\alpha \in \Delta_N$ with degree g matrices is denoted by $\mathbb{H}_{(g)}$ and the subset corresponding to the matrices with order $m \times n$ is denoted by $\mathbb{H}_{(g)}^{m\times n}$.

The relaxation proposed in Oliveira and Peres (2007) can be used in the parameter-dependent LMIs of Theorem 1 by forcing a homogeneous polynomial structure in the LMI variables $L(\alpha)$, $M(\alpha)$ and $F(\alpha)$, turning them into HPPD matrices. The relaxed condition is given by the LMIs stemming from the matrices coefficients of the HPPD matrices, and the LMI variables are the matrix coefficients of the monomials of the HPPD matrices. Whilst this procedure is systematic, it can become very complex as the degree g of the HPPD matrices increases. Nevertheless, the specialized parser ROLMIP [1] can be used to automatically carry this relaxation (Agulhari et al., 2012).

It is important to note that for the particular case that $g = 0$, the recuperation of the filter from the LMI variables using (12) is free from the parameter α, and consequently, the assumption that α can be read in real time is no longer required; the filter is robust in the sense that the parameter can be considered uncertain. The reason why is used degrees greater than zero is that a sequence of less conservative LMI relaxations may be obtained in the conditions of Theorem 1 by increasing the degree g, as will be clearer in the next theorems.

Theorem 2. For given $\bar{g}$ and $\bar{\mu}_2$, let $c_2^*(\bar{g})$ be the optimal solution of

$$\begin{gathered} \min \quad c_2 \\ \text{such that (11) holds with } g = \bar{g} \text{ and } \mu_2 = \bar{\mu}_2. \end{gathered}$$

Then, $c_2^*(\bar{g}+1) \leq c_2^*(\bar{g})$.

Proof. If there exist scalars μ_1,$\bar{\mu}_2$ and μ_3; matrix $K \in \mathbb{R}^{n\times n}$; and matrices W, Z, L, M, $F \in \mathbb{H}_{(g)}$ such that (11) holds, then μ_1, $\bar{\mu}_2$, μ_3, K and the following matrices

$$\left(\sum_{i=1}^{N}\alpha_i\right)W, \left(\sum_{i=1}^{N}\alpha_i\right)Z, \left(\sum_{i=1}^{N}\alpha_i\right)L, \left(\sum_{i=1}^{N}\alpha_i\right)M, \left(\sum_{i=1}^{N}\alpha_i\right)F,$$

belonging to $\mathbb{H}_{(g+1)}$, are a particular solution to (11), since $\alpha \in \Delta_N$. Hence, the minimization of c_2 subject to (11) for $\bar{g}+1$ produces at least the same optimal value obtained with $\bar{g}$, which implies that $c_2^*(\bar{g}+1) \leq c_2^*(\bar{g})$.

Theorem 3. For given $\bar{g}$, $\bar{\mu}_1$ and $\bar{\mu}_3$, let $c_1^*(\bar{g})$ and $d^*(\bar{g})$ be the optimal solution of problems

$$\begin{gathered} \max \quad c_1 \\ \text{such that (11) holds with } g = \bar{g} \text{ and } \mu_1 = \bar{\mu}_1, \end{gathered}$$

$$\begin{gathered} \max \quad d \\ \text{such that (11) holds with } g = \bar{g} \text{ and } \mu_3 = \bar{\mu}_3, \end{gathered}$$

respectively. Then $c_1^*(\bar{g}) \leq c_1^*(\bar{g}+1)$ and $d^*(\bar{g}) \leq d(\bar{g}+1)$.

Proof. Similar to Theorem 2.

The optimization problems in Theorem 2 and Theorem 3 can be seen as optimum filtering problems within FTB context. For example, to design a filter which rejects the maximum possible types of disturbances, one may try to maximize d.

The computational complexity of the LMIs is estimated by the number of scalar variables V and the number of LMI scalar rows L. For Theorem (1),

$$V = n(p+q+n)\,\mathrm{card}(\mathscr{S}_g) + n(n+1) + \frac{q(q+1)}{2} + 3, \tag{20}$$

$$L = (2n+r)\,\mathrm{card}\left(\mathscr{S}_{g+f+1}\right) + (4n+p+r)\,\mathrm{card}\left(\mathscr{S}_{g+f}\right) + n + 1. \tag{21}$$

By increasing the degree g, the number of decision variables is also increased and in consequence, the complexity of the LMIs also raises. However, by using an extension of Pólya's theorem (Oliveira and Peres, 2005, 2007), and based on the fact that the time-varying parameters α belong to the unit N-simplex, the conditions of Theorem (1) may also be improved using a sufficiently large positive integer f with no increase in the number of variables for a given degree g by multiplying the LMIs (11) by the factor $\left(\sum_{i=1}^{N}\alpha_i\right)^f$.

[1] Available for download at `http://www.dt.fee.unicamp.br/~agulhari/rolmip/rolmip.htm`.

Table 1. Minimum upper bounds of c_2 and maximum lower bounds of c_1 and d for different values of g and f.

Degree g	Index f	c_2	c_1	d
0	0	3.5381	-	0.5027
1	0	3.2441	-	0.5307
1	1	3.1724	0.0042	0.5307
1	2	3.1720	0.0042	0.5307
2	0	3.1511	0.0118	0.5308
2	1	3.0856	0.0164	0.5308
2	2	3.0856	0.0164	0.5308
3	0	3.0856	0.0164	0.5308
3	1	3.0656	0.0175	0.5309

4. NUMERICAL EXAMPLE

The numerical example was performed using the solver SeDuMi (Sturm, 1999) and the parsers YALMIP (Löfberg, 2004) and ROLMIP (Agulhari et al., 2012) within Matlab environment.

Consider the system (2) with matrices in polytope (3) with the following vertices

$$A_1 = \begin{bmatrix} -1.0 & 2.0 \\ -3.0 & -2.0 \end{bmatrix}, A_2 = \begin{bmatrix} -1.0 & 2.0 \\ -3.0 & -1.0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -2.0 & 2.0 \\ -3.0 & -2.0 \end{bmatrix}, A_4 = \begin{bmatrix} -2.0 & 2.0 \\ -3.0 & -1.0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.5 \\ 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}, B_3 = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}, B_4 = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix},$$

$$C_{y1} = [1.0\ 0.5], C_{y2} = [1.2\ 0.5],$$
$$C_{y3} = [1.0\ 0.6], C_{y4} = [1.2\ 0.6],$$

$$C_{z1} = [0.6\ 1.0], C_{z2} = [1.0\ 1.0],$$
$$C_{z3} = [0.6\ 1.2], C_{z4} = [1.0\ 1.2],$$
$$D_1 = D_2 = 0.1, D_3 = D_4 = 0.2$$

and the slowly varying parameter

$$\alpha(t) = \left(\tfrac{1}{2}\sin^2(\omega t), \tfrac{1}{2}\sin^2(\omega t), \tfrac{1}{2}\cos^2(\omega t), \tfrac{1}{2}\cos^2(\omega t)\right), \quad (22)$$

with ω sufficiently small. It is easy to check that (22) belongs to Δ_4 for all $t \geq 0$.

Theorems 2 and 3 along with Pólya's relaxation are applied with $\bar{\mu}_1 = 1$, $\bar{\mu}_2 = 1$, $\bar{\mu}_3 = 1$ and $\beta = 0.6$ in order to investigate the effect of increasing g and f in the search of minimum upper bounds of c_2 attained by the conditions of Theorem 2 and also in the search of maximum lower bounds of c_1 and d attained by the conditions of Theorem 3. The chosen FTB parameters for this example are $c_1 = 0.1$, $c_2 = 2$, $d = 1$, $T = 1.5$, $R_e = 4\mathbf{I}$ and $R_p = 4\mathbf{I}$. The results of the optimization problem are summarized in Table 1 (when the parameter is the value being optimized the corresponding value of the parameter should be ignored).

As can be seen in Table 1, the conditions of Theorem 3 are not able to provide a robust filter nor a LPV filter with a $c_1 > 0$. Also, by using degree $g = 3$ and $f = 1$ it was possible to obtain an upper bound to c_2 approximately 13.35% smaller than the robust filter corresponding to $g = 0$ and $f = 0$. Finally, as illustrated by the maximum lower bounds obtained by d, it may happen that the gain obtained increasing g and f is not appreciable. In this case, it would be better use the robust filter corresponding to $g = 0$ if existent.

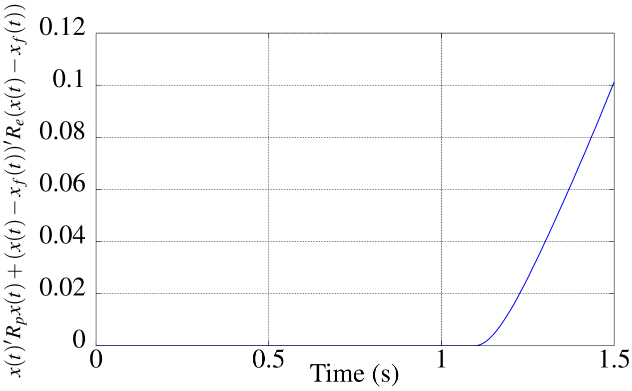

Figure 1. Sum of the weighted quadratic norm of the states of the plant and observation error.

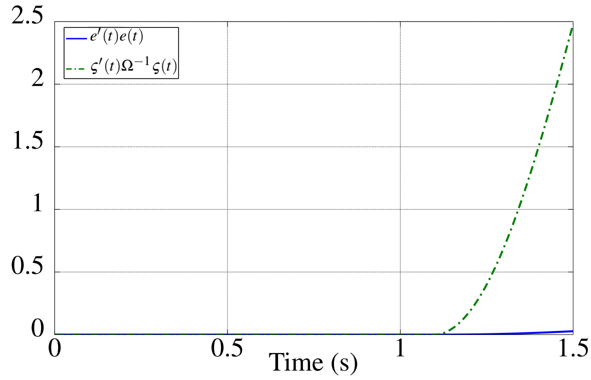

Figure 2. Time simulation comparing the norm of the error with the bound imposed by Ω.

Consider now system (2) with the varying parameter (22) with $\omega = 0.001$. We wish to check if the FTB filter obtained by Theorem 2 with $\bar{\mu}_2 = 1$, $g = 3$ and $f = 1$ satisfies the FTB condition with respect to $(0.1, 3.06, \mathscr{W}_1, 1.5, 4\mathbf{I}, 4\mathbf{I})$ and the Ω-bound condition defined in (8) considering a disturbance given by the step function

$$w(t) = \begin{cases} 0, & \text{if } t \leq 1.1s, \\ 0.9, & \text{if } t > 1.1s, \end{cases}$$

which represents the worst type of signal belonging to the class $\mathscr{W}_1$. Since this filter is obtained using the assumption that α is a slowly varying parameter, one must verify if $\dot{\alpha}$ is really sufficiently small by time-domain simulations. Considering zero initial conditions, a time-simulation was performed in the time interval $t \in [0, 1.5s]$.

As shown in Figure 1, the designed filter satisfies the FTB condition and

$$\max_{t\in[0,1.5s]} \left\{ \begin{bmatrix} x(t) \\ x(t) - x_f(t) \end{bmatrix}' \begin{bmatrix} R_p & \mathbf{0} \\ \mathbf{0} & R_e \end{bmatrix} \begin{bmatrix} x(t) \\ x(t) - x_f(t) \end{bmatrix} \right\} = 0.06,$$

which is approximately 2% of the value of $c_2 = 3.0653$. Moreover, as can be seen in Figure 2, the estimation error of the filter also satisfies the Ω-bound condition defined in (8). The tracking error is shown in Figure 3. Although the choice of Γ_{22} did not nullify the error between z and z_f, it ensured a maximum error of 0.0764. The difficulty of having a null estimation error in a finite time horizon is due to the small time that the filter has to dynamically estimate the output z.

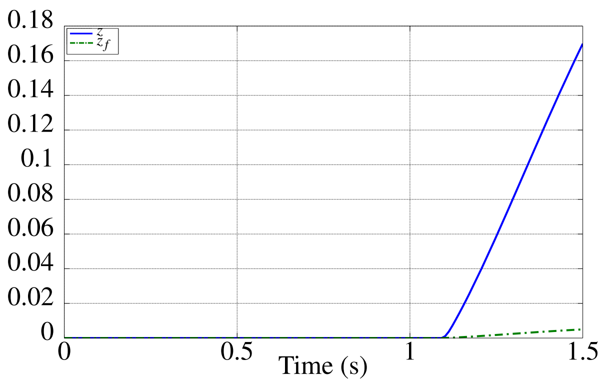

Figure 3. Time simulation of the outputs z and z_f.

5. CONCLUSION

This paper considers the problem of filter design for LPV continuous-time systems. The filter was obtained under the assumption that the parameter of the LPV system is sufficiently slow time-varying. If this assumption is satisfied the obtained filter guarantees that the augmented system is bounded during a finite time horizon under the presence of bounded disturbances. The design conditions are represented by a LMI feasibility problem, which can be relaxed via homogeneous polynomials techniques and Pólya's theorem. It was shown that a sequence of less conservative conditions may be obtained by increasing the degree g of the HPPD matrices or increasing the positive integer f based on Pólya's theorem. To handle the algebraic manipulation involved in the construction of the relaxation of the parameter-dependent LMIs, the specialized parser ROLMIP was used, easing the work.

REFERENCES

Agulhari, C.M., de Oliveira, R.C.L.F., and Peres, P.L.D. (2012). Robust LMI Parser: a computational package to construct LMI conditions for uncertain systems. In *XIX Brazilian Conference on Automation (CBA 2012)*, 2298–2305. Campina Grande, PB, Brazil.

Amato, F., Ariola, M., and Cosentino, C. (2005). Finite-time control of linear time-varying systems via output feedback. In *Proceedings of the 2005 American Control Conference*, 4722–4726. Portland, OR, USA.

Amato, F., Ariola, M., Cosentino, C., Abdallah, C.T., and Dorato, P. (2003). Necessary and sufficient conditions for finite-time stability of linear systems. In *Proceedings of the 2003 American Control Conference*, volume 5, 4452 – 4456.

Amato, F., Ariola, M., and Dorato, P. (2001). Finite-time control of linear systems subject to parametric uncertainties and disturbances. *Automatica*, 37(9), 1459–1563.

Amato, F., Ambrosino, R., Ariola, M., Cosentino, C., and De Tommasi, G. (2014). *Finite-Time Stability and Control*. Springer, London, UK.

Baslamisli, S.Ç., Köse, İ.E., and Anlaş, G. (2009). Gain-scheduled integrated active steering and differential control for vehicle handling improvement. *Vehicle System Dynamics*, 47(1), 99–119.

Bliman, P.A., Oliveira, R.C.L.F., Montagner, V.F., and Peres, P.L.D. (2006). Existence of homogeneous polynomial solutions for parameter-dependent linear matrix inequalities with parameters in the simplex. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 1486–1491. San Diego, CA, USA.

Borges, R.A., Ishihara, J.Y., Rocha, I.G., Risso, L.O., and Kussaba, H.T.M. (2013). Finite time robust filtering for time-varying uncertain polytopic linear systems. In *Proceedings of the 2013 European Control Conference*, 1854–1859. Zurich, Switzerland.

Chilali, M. and Gahinet, P. (1996). $\mathscr{H}_\infty$ design with pole placement constraints: an LMI approach. *IEEE Transactions on Automatic Control*, 41(3), 358–367.

Ganguli, S., Marcos, A., and Balas, G. (2002). Reconfigurable LPV control design for Boeing 747-100/200 longitudinal axis. In *Proceedings of the 2002 American Control Conference*, volume 5, 3612–3617. IEEE.

He, S. and Liu, F. (2011). Robust L_2-L_∞ filtering of time-delay jump systems with respect to the finite-time interval. *Mathematical Problems in Engineering*. http://www.hindawi.com/journals/mpe/2011/839648/.

Khalil, H.K. (2002). *Nonlinear Systems*. Prentice Hall, Upper Saddle River, NJ, 3rd edition.

Lin, H.H., Beck, C., and Bloom, M. (2008). Multivariable LPV control of anesthesia delivery during surgery. In *Proceedings of the 2008 American Control Conference*, 825–831. IEEE.

Liu, C., Zhang, Y., and Sun, H. (2012). Finite-time H_∞ filtering for singular stochastic systems. *Journal of Applied Mathematics*. http://www.hindawi.com/journals/jam/2012/615790/.

Löfberg, J. (2004). YALMIP: A toolbox for modeling and optimization in MATLAB. In *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control Systems Design*, 284–289. Taipei, Taiwan. http://control.ee.ethz.ch/~joloef/yalmip.php.

Luan, X., Liu, F., and Shi, P. (2010). Finite-time filtering for non-linear stochastic systems with partially known transition jump rates. *IET Control Theory & Applications*, 4(5), 735–745.

Oliveira, R.C.L.F. and Peres, P.L.D. (2005). Stability of polytopes of matrices via affine parameter-dependent Lyapunov functions: Asymptotically exact LMI conditions. *Linear Algebra and Its Applications*, 405, 209–228.

Oliveira, R.C.L.F. and Peres, P.L.D. (2007). Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations. *IEEE Transactions on Automatic Control*, 52(7), 1334–1340.

Peña, R.S.S. and Ghersin, A.S. (2010). LPV control of glucose for diabetes type I. In *Engineering in Medicine and Biology Society (EMBC), 2010 Annual International Conference of the IEEE*, 680–683. IEEE.

Shamma, J.S. (2012). An overview of LPV systems. In *Control of Linear Parameter Varying Systems with Applications*, 3–26. Springer.

Sturm, J.F. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1–4), 625–653. http://sedumi.ie.lehigh.edu/.

Toth, R. (2010). *Modeling and Identification of Linear Parameter-Varying Systems*. Lecture Notes in Control and Information Sciences. Springer.