Published in 2015 54th IEEE Conference on Decision and Control (CDC). DOI:10.1109/CDC.2015.7403370

Uniform versions of Finsler's lemma

H. T. M. Kussaba, J. Y. Ishihara, R. A. Borges

***Abstract*— Finsler's lemma and its variants are used extensively in control and optimization literature. Although there are several variants of this lemma, it can be summarized as an equivalence of a quadratic inequality with a linear constraint to a linear matrix inequality (LMI) with possibly more extra variables than the original problem. When applying Finsler's lemma to problems that depends of a parameter, the extra variables stemming from the use of Finsler's lemma can also be dependent of this parameter. In this paper it is investigated when these extra variables can assume a simple structure without loss of generality and some sufficient conditions that guarantees this assumption are given. These sufficient conditions can be applied when investigating parameter dependent LMIs that emerge, for instance, when dealing with the robust stability of systems with polytopic uncertainty.**

***Index Terms*— Finsler's lemma, Linear matrix inequality, Robust stability**

I. INTRODUCTION

Some problems in system and control theory can frequently be recast as a feasibility test asking the existence of a positive definite (or negative definite) matrix satisfying a constraint—for instance, the classical Lyapunov's theorem [1] says that the asymptotically stability of a linear system

$$\dot{x}(t) = Ax(t) \tag{1}$$

is equivalent to the existence of a symmetric matrix P satisfying the inequality

$$\begin{bmatrix} P & 0 \\ 0 & -(A^T P + PA) \end{bmatrix} \succ 0, \tag{2}$$

where the symbol $\succ$ denotes that the above matrix is positive definite. Restating the above problem in a more abstract way, the problem is equivalent to find a matrix X such that

$$F(X) \succ 0, \tag{3}$$

where $F : \mathbb{R}^{m\times n} \to \mathbb{S}^{p\times p}$ is a function of the set of real matrices $m \times n$ to the set of symmetric real matrices of order $p \times p$. In particular, when F is affine, that is, $F(X) - F(0)$ is linear in X, this type of problem is known as a linear matrix inequality (LMI). Besides the stability criterion for linear system given in (2), several other analysis and design problems of control systems can be established via the LMI approach, such as controllability, reachability, observability, detectability, $\mathscr{H}^2$ and $\mathscr{H}^\infty$ performance analysis, as well control and filter synthesis [2]–[4]. It should also be noted that there are several computational packages specialized in efficiently solving LMIs, to name a few: LMI Control Toolbox [5], SeDuMi [6] and SDPT3 [7] for MATLAB, CSDP [8] for the C programming language and SDPA [9] for the C++ programming languages.

When the matrices of the control system, however, has dependence on a parameter (such as the case of uncertain and linear parameter varying systems (LPV) [10], for instance), the LMIs also become dependent on this parameter and their feasibility is possibly an infinite dimensional problem. For example, if the matrix A in (1) was not precisely known but had an uncertainty $\alpha \in S$, one must had to check (2) for every $\alpha \in S$, i.e. it would be necessary to find a matrix valued function $P : S \to \mathbb{S}^{n\times n}$ satisfying

$$\begin{bmatrix} P(\alpha) & 0 \\ 0 & -A^T(\alpha)P(\alpha) - P(\alpha)A(\alpha) \end{bmatrix} \succ 0, \ \forall \alpha \in S. \tag{4}$$

The inequality (4) is the first example of a parameter dependent LMI (PD-LMI). Depending on the properties of the set S, the problem is not computationally tractable: for instance, if S is infinite, it is known that a generic PD-LMI is NP-hard [11]. Nevertheless, there have been several approaches in the linear matrix inequality literature to relax PD-LMIs problems. For instance, a naive method to solve (4) would be to approximate S by a finite set of points—this would relax the PD-LMI into a finite number of LMIs. However, this method usually requires a great number of points to its results become satisfactory, which generates a heavy computational load. Moreover, if some points are excluded it is possible that the result turns out to be excessively optimist [12].

An alternative relaxation procedure to this method is using a polynomial relaxation. This relaxation procedure consists in restricting the matrix valued functions that are variables of the PD-LMI to matrix polynomials functions of a fixed degree g and using the matrix coefficients of the polynomial stemmed of this procedure to generates new LMIs independent of the parameter that imply the original PD-LMI condition. With the increase of the degree g of the polynomial, less conservative sets of conditions can be found that imply the original PD-LMI. In fact, in [13] it is proved that if a PD-LMI has a solution, S is compact and the LMI depends continuously in the parameter, then without loss of generality, there exists a solution that is a polynomial matrix for this PD-LMI. In the particular case that S is a unit-simplex, there will also be, without loss of generality, a solution that is a homogeneous polynomial matrix [14].

Along with the technique of polynomial relaxation, one may also use Finsler's lemma to achieve less conservativeness by the introduction of extra variables to the LMI

The authors are with the Department of Electrical Engineering, University of Brasília – UnB, 70910-900, Brasília, DF, Brazil. Currently, J. Y. Ishihara is a visiting associate researcher at Univ. California Los Angeles (UCLA). kussaba@lara.unb.br, ishihara@ene.unb.br, raborges@ene.unb.br.

[15]. For instance, in the context of robust stability, PD-LMI conditions which contains (4) can be formulated by using Finsler's lemma [16]. In [17], this technique is used for investigating the quadratic stabilizability of Takagi-Suzeno fuzzy systems.

In all of the aforementioned papers, the extra variables introduced by Finsler's lemma are parameter dependent. The use of parameter dependent variables are applied to lessen the conservatism at the cost of increasing computational burden. However, there are some cases in which a parameter independent variable is as good as a parameter dependent variable indicating that there is no gain to impose a structure to the extra variable as shown by the main results proposed in this paper.

Notation: In the sequel the following notation will be used: $\mathbb{N}$ is the set of non-negative integer numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^{m\times n}$ the set of real matrices of order $m\times n$, $\mathbb{S}^n$ the set of symmetric real matrices of order $n\times n$, $\mathbb{S}^n_+$ the set of symmetric positive definite matrices of order $n\times n$. $N\prec 0$ indicates that N is a symmetric negative definite matrix. A^T denotes the transpose of matrix A.

II. Main results

Finsler's lemma is one of the most important tools in the control literature. It is used to eliminate variables from matrix inequalities in order to turn the problem into a simpler form, or to introduce slack variables in order to lessen conservativeness [16], [18].

The assertion of the Finsler's lemma is the equivalence between a statement about a quadratic inequality with linear constraint and another statements about the negative definiteness of a matrix as stated below.

Lemma 1. *[19] Let $x\in\mathbb{R}^n$, $Q\in\mathbb{S}^n$ and $B\in\mathbb{R}^{m\times n}$, with $\operatorname{rank}(B)<n$. Then the following statements are equivalent:*

i) *$x^TQx<0$ for all $x\in\mathbb{R}^n$ such that $x\neq 0$ and $Bx=0$.*
ii) *There exists $\mu\in\mathbb{R}$ such that $Q-\mu B^TB\prec 0$.*

The importance of this lemma can be noted by the existence of several variants along the literature [15] and by the fact that it is equivalent to other important results in control literature such as Yakubovich's S-lemma [20]. In fact, Finsler's lemma have been proved several times; for proofs of it the reader is referred to the papers [15], [21]–[24].

When treating uncertain systems, the matrices Q or B can become dependent of a parameter and a pointwise extension of Finsler's lemma is easily obtained as stated in Lemma 2.

Lemma 2. *Let $S\subseteq\mathbb{R}$, $x\in\mathbb{R}^n$, $Q:S\to\mathbb{S}^n$ and $B:S\to\mathbb{R}^{m\times n}$, with $\operatorname{rank}(B(s))<n$ for all $s\in S$. Then the following statements are equivalent:*

(F1) *For each $s\in S$, one has $x^TQ(s)x<0$ for all $x\in\mathbb{R}^n$ such that $x\neq 0$ and $B(s)x=0$.*

(F2) $(\forall s\in S)\,(\exists\mu(s)\in\mathbb{R}):Q(s)-\mu(s)B^T(s)B(s)\prec 0.$

Proof: Follows directly from extending Lemma 1 pointwisely. ■

In this context, the main goal of this paper is to investigate if $(F2)$ is also valid uniformly in μ, that is, if

(F3) $(\exists\bar{\mu}\in\mathbb{R})\,(\forall s\in S):Q(s)-\bar{\mu}B^T(s)B(s)\prec 0.$

It is obvious that $(F3)$ implies $(F2)$, however as shown in the following counterexample, the uniformity in the extra variable μ is not true in general and extra hypothesis are necessary to assure that $(F2)$ implies $(F3)$.

Example 1. Let $Q:S\to\mathbb{R}^{2\times 2}$, $B:S\to\mathbb{R}^{1\times 2}$ and $S\subset\mathbb{R}$ be given by

$$Q(s)=\begin{bmatrix}-1 & 0\\ 0 & s\end{bmatrix},\; B^T(s)=\begin{bmatrix}0\\ s\end{bmatrix}\text{ and } S=(0,1].$$

One has that

$$Q(s)-\mu(s)B^T(s)B(s)=\begin{bmatrix}-1 & 0\\ 0 & s-\mu(s)s^2\end{bmatrix}\prec 0,$$

which is valid if and only if $s-\mu(s)s^2<0$. Since $0\notin S$, it follows that $1/s<\mu(s)$.

Thus, to grant that $(F2)$ will be satisfied one can take $\mu(s)=\frac{1}{s}+\varepsilon$ with $\varepsilon>0$. However, $(F3)$ can not be guaranteed since the function $1/s$ grows without bound as s goes to 0. □

The first result of this paper, a sufficient condition to guarantee the uniformity in μ, is given next in Lemma 3.

Lemma 3. *Let $Q:S\to\mathbb{S}^n$ and $B:S\to\mathbb{R}^{m\times n}$ be functions on $S\subseteq\mathbb{R}$ such that*

$$\sup_{s\in S}\inf\left\{\mu\in\mathbb{R}\mid Q(s)-\mu B^T(s)B(s)\prec 0\right\}<\infty. \tag{5}$$

Then the statements $(F2)$ and $(F3)$ are equivalent.

Proof: The proof that $(F3)$ implies $(F2)$ is immediate. We now prove that $(F2)$ implies $(F3)$. For each $s\in S$, define

$$\mathscr{M}(s)=\left\{\mu\in\mathbb{R}\mid Q(s)-\mu B^T(s)B(s)\prec 0\right\}.$$

By $(F2)$, one has that $\mathscr{M}(s)\neq\emptyset$. One also has that if $\mu^\star\in\mathscr{M}(s)$, then $(\mu^*+\alpha)\in\mathscr{M}(s)$ for all $\alpha\geq 0$, since

$$\begin{aligned}0\succ&Q(s)-\mu^*B^T(s)B(s)\\ \succeq&Q(s)-\mu^*B^T(s)B(s)-\underbrace{\alpha B^T(s)B(s)}_{\succeq 0}\\ =&Q(s)-(\mu^*+\alpha)B^T(s)B(s).\end{aligned}$$

Therefore, there always exists a $\mu^*\in\mathscr{M}(s)$ such that $[\mu^*,\infty)\subseteq\mathscr{M}(s)$.

By (5), there exists $m\in\mathbb{R}$ such that

$$\inf\left\{\mu\in\mathbb{R}\mid Q(s)-\mu B^T(s)B(s)\prec 0\right\}<m,\ \forall s\in S.$$

Consequently, $m\in\mathscr{M}(s)$ for all $s\in S$ and $\bar{\mu}=m$ is such that $(F3)$ holds. ■

It is important to note that although Lemma 3 is very general and does not require any special structure on the functions Q and B nor in the set S, it is very hard to check (5).

Nevertheless, Lemma 3 can be used together with conditions on the functions Q and B for derive simpler criteria to the uniformity of μ as it is done in the following Theorem 1.

Theorem 1. *Let $Q: S \to \mathbb{S}^n$ and $B: S \to \mathbb{R}^{m\times n}$ be continuous functions on a compact $S \subset \mathbb{R}$. If*

$$\operatorname{Ker} B(s) = K,\ \forall s \in S, \tag{6}$$

then the statements $(F2)$ and $(F3)$ are equivalent.

Proof: The proof that $(F3)$ implies $(F2)$ is immediate. We now prove that $(F2)$ implies $(F3)$. Let $k := \dim K$. One has that $\operatorname{rank} B(s)$ is independent of s since

$$\operatorname{rank} B(s) = n - \dim K = n - k.$$

Consider an orthogonal decomposition of $\mathbb{R}^n$ given by

$$\mathbb{R}^n = K^{\perp} \oplus K$$

where $\oplus$ denotes the direct sum of vector spaces. Consider also a block matrix $\begin{bmatrix} C_1 & C \end{bmatrix}$ such that the columns of C_1 form a basis for $K^{\perp}$ and the columns of C form a basis for K. It follows that

$$B(s)\begin{bmatrix} C_1 & C \end{bmatrix} = \begin{bmatrix} B_1(s) & 0 \end{bmatrix},$$

where $B_1(s) \in \mathbb{R}^{m\times(n-k)}$ for all $s \in S$.

One has that $B_1(s)$ is full column rank. In fact,

$$\begin{aligned} \operatorname{rank} B_1(s) &= \operatorname{rank} \begin{bmatrix} B_1(s) & 0 \end{bmatrix} \\ &= \operatorname{rank} B(s)\begin{bmatrix} C_1 & C \end{bmatrix} \\ &= \operatorname{rank} B(s) \\ &= n-k, \end{aligned}$$

where the third equality follows since $\begin{bmatrix} C_1 & C \end{bmatrix}$ is full rank.

Thus, $R(s) := B_1^T(s) B_1(s)$ is invertible for each $s \in S$. One also has that $R(s)$ is continuous on S because since $B(s)$ is continuous, $B_1(s) = B(s) C_1$ is also continuous on S.

By congruence transform, one has that

$$Q(s) - \mu(s) B^T(s) B(s) \prec 0$$

if and only if

$$\begin{bmatrix} C_1^T \\ C^T \end{bmatrix} \left(Q(s) - \mu(s) B^T(s) B(s)\right) \begin{bmatrix} C_1 & C \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11}(s) - \mu(s) B_1^T(s) B_1(s) & \bar{Q}_{12}(s) \\ \bar{Q}_{12}^T(s) & \bar{Q}_{22}(s) \end{bmatrix} \prec 0, \tag{7}$$

where $\bar{Q}_{11}(s) = C_1^T Q(s) C_1$, $\bar{Q}_{12}(s) = C_1^T Q(s) C$ and $\bar{Q}_{22}(s) = C^T Q(s) C$.

Applying a Schur transform, (7) is equivalent to $\bar{Q}_{22} \prec 0$ and

$$\bar{Q}_{11}(s) - \bar{Q}_{12}(s)\bar{Q}_{22}^{-1}(s)\bar{Q}_{12}^T(s) - \mu(s) B_1^T(s) B_1(s) \prec 0.$$

Consequently, defining

$$\hat{Q}(s) = \bar{Q}_{11}(s) - \bar{Q}_{12}(s)\bar{Q}_{22}^{-1}(s)\bar{Q}_{12}^T(s)$$

one has that

$$\left\{\mu \in \mathbb{R} \mid Q(s) - \mu(s) B^T(s) B(s) \prec 0\right\} = \left\{\mu \in \mathbb{R} \mid \hat{Q}(s) - \mu(s) B_1^T(s) B_1(s) \prec 0\right\}.$$

Since Q is continuous, it follows that $\bar{Q}_{11}$, $\bar{Q}_{12}$ and $\bar{Q}_{22}$ are continuous. As a result, $\hat{Q}$ is also continuous. Let $R^{\frac{1}{2}}(s)$ be a matrix square root of $R(s) := B_1^T(s) B_1(s)$. Since B_1^T is continuous, $R^{\frac{1}{2}}$ is also continuous.

Also, one has that

$$\hat{Q}(s) - \mu(s) B_1^T(s) B_1(s) \prec 0$$

if and only if

$$R^{-\frac{1}{2}}(s)\hat{Q}(s)R^{-\frac{1}{2}}(s) - \mu(s) I \prec 0.$$

Defining $Q_1^*(s) := R^{-\frac{1}{2}}(s)\hat{Q}(s)R^{-\frac{1}{2}}(s)$ it follows that Q_1^* is continuous and that

$$\begin{aligned} \inf\{\mu \in \mathbb{R} \mid Q_1^*(s) - \mu I \prec 0\} &= \inf\{\mu \in \mathbb{R} \mid Q_1^*(s) \prec \mu I\} \\ &= \inf\{\mu \in \mathbb{R} \mid \lambda_{max}[Q_1^*(s)] < \mu\} \\ &= \lambda_{max}[Q_1^*(s)]. \end{aligned}$$

Since S is compact and the function λ_{max} is continuous, one has by Weierstrass' theorem that

$$\sup_{s\in S} \inf\left\{\mu \in \mathbb{R} \mid Q(s) - \mu B^T(s) B(s) \prec 0\right\} = \sup_{s\in S}\{\lambda_{max}[Q_1^*(s)]\} < \infty.$$

By Lemma 3, the result follows. ■

It is important to note that Theorem 1 does not assume any requirement in the function $\mu(s)$ and this function can even be discontinuous.

In counterpart, if the function $\mu(s)$ is known to be continuous then the requirement of continuity of Q and B can be dropped. This is shown in Theorem 2.

Theorem 2. *Let $Q: S \to \mathbb{S}^n$ and $B: S \to \mathbb{R}^{m\times n}$ be matrix valued functions on a compact $S \subset \mathbb{R}$. Then the following statements are equivalent:*

i) *There exists a continuous function $\mu: S \to \mathbb{R}$ such that $Q(s) - \mu(s)B^T(s)B(s) \prec 0$ for every $s \in S$.*
ii) *There exists $\bar{\mu} \in \mathbb{R}$ such that $Q(s) - \bar{\mu}B^T(s)B(s) \prec 0$ for every $s \in S$.*

Proof: If ii) is valid, then $\mu(s) = \bar{\mu}$ satisfies i). Suppose now that i) is valid. By Weierstrass' theorem, the function μ has an maximum m in S. Thus

$$0 \succ Q(s) - \mu(s)B^T(s)B(s) \succeq Q(s) - mB^T(s)B(s)$$

and ii) is satisfied with $\bar{\mu} = m$. ■

The next Lemma 4 is an auxiliary result to give an useful criterion to check for the existence of a continuous μ .

Lemma 4. *Let $M \in \mathbb{S}^n$ and $N \in \mathbb{S}_+^n$. Then exists $\mu \in \mathbb{R}$ such that*

$$M - \mu N \prec 0.$$

In fact, one such μ is given by

$$\mu = \frac{\lambda_{max}(M) + |\lambda_{max}(M)|}{\lambda_{min}(N)} + 1.$$

Proof: Since $N \succ 0$, one has that

$$0 \prec \lambda_{min}(N) I \preceq N. \tag{8}$$

Moreover, the fact that $M \in \mathbb{S}^n$ yields

$$\lambda_{min}(N)M \preceq \lambda_{max}(M)\lambda_{min}(N)I. \quad (9)$$

Since $\lambda_{max}(M) \in \mathbb{R}$ and $\lambda_{min}(N) > 0$, exists $\eta > 0$ such that

$$\lambda_{max}(M) + \eta\lambda_{min}(N) > 0. \quad (10)$$

In fact, it is enough to take

$$\eta = \frac{|\lambda_{max}(M)|}{\lambda_{min}(N)} + 1.$$

From $\eta > 0$ and $\lambda_{min}(N) > 0$ it follows that

$$\lambda_{max}(M) < \lambda_{max}(M) + \eta\lambda_{min}(N)$$

and

$$\lambda_{max}(M)\lambda_{min}(N) < [\lambda_{max}(M) + \eta\lambda_{min}(N)]\,\lambda_{min}(N). \quad (11)$$

Inequalities (9) and (11) yields

$$\begin{aligned} \lambda_{min}(N)M &\prec \lambda_{min}(N)\,[\lambda_{max}(M) + \eta\lambda_{min}(N)]\,I \\ &\preceq [\lambda_{max}(M) + \eta\lambda_{min}(N)]\,N. \end{aligned}$$

Since $\lambda_{min}(N) \succ 0$,

$$M - \left[\frac{\lambda_{max}(M) + \eta\lambda_{min}(N)}{\lambda_{min}(N)}\right] N \prec 0.$$

Therefore it is enough to take

$$\mu = \frac{\lambda_{max}(M) + \eta\lambda_{min}(N)}{\lambda_{min}(N)} = \frac{\lambda_{min}(N) + \lambda_{max}(M) + |\lambda_{max}(M)|}{\lambda_{min}(N)}.$$

■

Corollary 1. *Consider $Q : S \to \mathbb{S}^n$ and $B : S \to \mathbb{R}^{m\times n}$ be continuous matrix valued functions on $S \subseteq \mathbb{R}$ such that $B(s)$ is full column rank for every $s \in S$. Then there exists $\mu : S \to \mathbb{R}$ continuous such that*

$$Q(s) - \mu(s)B^T(s)B(s) \prec 0, \ \forall s \in S.$$

One such function μ is

$$\mu(s) = \frac{\lambda_{max}(Q(s)) + |\lambda_{max}(Q(s))|}{\lambda_{min}(B^T(s)B(s))} + 1.$$

Proof: Follows directly from Lemma 4 and from the fact that λ_{max} and λ_{min} are continuous functions on their arguments. ■

The following corollary from Theorem 2 is important in the context of polynomial relaxation procedures for PD-LMIs [13], [14], wherein is assumed that μ is a polynomial function.

Corollary 2. *Let $Q : S \to \mathbb{S}^n$ and $B : S \to \mathbb{R}^{m\times n}$ be matrix valued functions on a compact $S \subset \mathbb{R}$. Then the following statements are equivalent:*

i) *There exists a polynomial function $\mu(s)$ such that $Q(s) - \mu(s)B^T(s)B(s) \prec 0$ for every $s \in S$.*
ii) *For every $s \in S$ there exists $\bar{\mu} \in \mathbb{R}$ such that $Q(s) - \bar{\mu}B^T(s)B(s) \prec 0$.*

If $S = \mathbb{R}^d$, it is possible to assume without loss of generality that μ is a rational function without singularities (for $d = 1$, this is equivalent to a polynomial function) with extra assumptions in the structure of the functions Q and B. Precisely, Q and B must be matrix polynomials, which is defined in Definition 1.

Definition 1. A **matrix polynomial** H of degree g is a function from $S \subseteq \mathbb{R}^d$ to $\mathbb{R}^{m\times n}$ that for $s = (s_1, \ldots, s_d) \in \mathbb{R}^d$ can be expressed as

$$H(s) = \sum_{\beta \in I^{(g)}} H_{\beta_1,\ldots,\beta_d} s_1^{\beta_1} \cdots s_d^{\beta_d},$$

where $H_{\beta_1,\ldots,\beta_d} \in \mathbb{R}^{m\times n}$ are the **matrices coefficients** of H and

$$I^{(g)} = \left\{ \beta = (\beta_1, \ldots \beta_d) \in \mathbb{N}^d \;\middle|\; \sum_{i=1}^{d} \beta_i \le g \right\}.$$

The set of all matrix polynomials will be denoted as $\mathbb{R}^{m\times n}(\mathbb{R}[s_1, \ldots, s_d])$. In particular, the set of matrix polynomials which takes values on symmetric matrices of order n will be denoted as $\mathbb{S}^n(\mathbb{R}[s_1, \ldots, s_d])$.

Theorem 3 below states that if Q and B are matrix polynomials and $S = \mathbb{R}^d$ then the existence of a pointwise solution for $(F2)$ is enough to assure the existence of a rational function without singularities satisfying the inequality in $(F2)$. This theorem is a particular case of a more general result of [25].

Theorem 3. *If $Q \in \mathbb{S}^n(\mathbb{R}[s_1, \ldots, s_d])$ and $B \in \mathbb{R}^{m\times n}(\mathbb{R}[s_1, \ldots, s_d])$ are matrix polynomials, then the following statements are equivalent:*

i) *For each $s \in \mathbb{R}^n$ there exists $\mu(s) \in \mathbb{R}$ such that $Q(s) - \mu(s)B^T(s)B(s) \prec 0$.*
ii) *There exists a rational function $\mu : \mathbb{R}^d \to \mathbb{R}$ without singularities such that $Q(s) - \mu(s)B^T(s)B(s) \prec 0$ for every $s \in \mathbb{R}^d$.*

Proof: For the proof, the following result from [25] will be used: suppose $G \in \mathbb{S}^n(\mathbb{R}[s_1, \ldots, s_d])$ is such that $G(a)$ is negative semidefinite for every a outside some ball in $\mathbb{R}^d$. Then for every $F \in \mathbb{S}^n(\mathbb{R}[s_1, \ldots, s_d])$ the following statements are equivalent:

1) For every $a \in \mathbb{R}^d$ there exists $r(a) \in \mathbb{R}$ such that $F(a) - r(a)G(a) \succ 0$.
2) There exists a rational function $r(x)$ without singularities such that $F(a) - r(a)G(a) \succ 0$ for every $a \in \mathbb{R}^d$.

The statement of theorem now follows directly by choosing $Q = -F$ and $G = -B^T B \preceq 0$. ■

The sufficient conditions proposed in this section enable the uniformity of the extra variable introduced by the application of Finsler's lemma in the context of parameter dependent matrices. In the next section, we highlight some situations where nothing is gained imposing additional structures in the extra scalar variable, since one can always choose it as a constant variable for all parameters of the involved dependent matrices.

III. Applications

In this section, some examples of applications of the results of Section II will be given.

Example 2. Consider an uncertain continuous-time linear system

$$\dot{x}(t) = A(\alpha)x(t), \tag{12}$$

where

$$A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i$$

and the value of the parameter $\alpha \in \mathbb{R}^N$ is unknown, but belonging to the unit N-simplex

$$\Delta_N = \left\{ \theta \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} \theta_i = 1,\ \theta_i \geq 0 \right\}.$$

Proceeding as in [15] and using Finsler's lemma (Lemma 2) with

$$Q(\alpha) = \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix},\ B^T(\alpha) = \begin{bmatrix} A^T(\alpha) \\ -I \end{bmatrix},$$

it is possible to prove that one sufficient condition to the robust stability of (12) is the existence of a positive definite matrix $P \in \mathbb{S}^n_+$ and a scalar function $\mu : \Delta_N \to \mathbb{R}$ such that

$$\begin{bmatrix} -\mu(\alpha)A^T(\alpha)A(\alpha) & \mu(\alpha)A^T(\alpha) + P \\ \mu(\alpha)A(\alpha) + P & -\mu(\alpha)I \end{bmatrix} \prec 0 \tag{13}$$

holds for all $\alpha \in \Delta_N$.

By Corollary 2, there is no gain to assume that μ is a polynomial function, because if there exists a polynomial μ satisfying (13), then there will also be a $\bar{\mu} \in \mathbb{R}$ such that

$$\begin{bmatrix} -\bar{\mu}A^T(\alpha)A(\alpha) & \bar{\mu}A^T(\alpha) + P \\ \bar{\mu}A(\alpha) + P & -\bar{\mu}I \end{bmatrix} \prec 0 \tag{14}$$

is satisfied.

Moreover, one may check if

$$\operatorname{Ker} \begin{bmatrix} A(\alpha) & -I \end{bmatrix}$$

is constant with relation to parameter α. If this is true, then by Theorem 1, the verification of (13) reduces to the verification of (14).

Example 3. Consider again the polytopic system from Example 2.

Using a variant of Finsler's lemma and proceeding as in [15], it is possible to show that another sufficient condition for the robust stability of (12) is the existence of a positive definite matrix $P \in \mathbb{S}^n_+$ and $n \times n$ matrix valued functions $F(\alpha)$ and $G(\alpha)$ such that

$$\begin{bmatrix} A^T(\alpha)F^T(\alpha) + F(\alpha)A(\alpha) & A^T(\alpha)G^T(\alpha) - F(\alpha) + P \\ G(\alpha)A(\alpha) - F^T(\alpha) + P & -G(\alpha) - G^T(\alpha) \end{bmatrix} \prec 0 \tag{15}$$

holds for all $\alpha \in \Delta_N$. The variables $F(\alpha)$ and $G(\alpha)$ can be seen as slack variables that augments the search space when compared with (13).

Following the steps of [15] to prove that (15) is equivalent to (13) in the case that the matrices are precisely known, we now prove the equivalence of (15) and (13) in the case of parameter dependent matrices by introducing additional hypothesis.

Considering that

$$X(\alpha) = \begin{bmatrix} F(\alpha) \\ G(\alpha) \end{bmatrix},$$

then (13) can be written as

$$Q(\alpha) - \mu(\alpha)B^T(\alpha)B(\alpha) \prec 0, \tag{16}$$

and (15) can be written as

$$Q(\alpha) + X(\alpha)B(\alpha) + B^T(\alpha)X^T(\alpha) \prec 0. \tag{17}$$

Multiplying (17) by left by $\left(B^{\perp}\right)^T(\alpha)$ and by right by $\mathscr{B}^{\perp}(\alpha)$ where

$$B^{\perp}(\alpha) = \begin{bmatrix} I \\ A(\alpha) \end{bmatrix}$$

one has that (17) implies

$$\left(B^{\perp}\right)^T(\alpha)Q(\alpha)B^{\perp}(\alpha) \prec 0. \tag{18}$$

Consider now a rank decomposition of $B(\alpha)$ given by $B_l(\alpha)B_r(\alpha)$ and define

$$D(\alpha) = B_r^T(\alpha)\left[B_r(\alpha)B_r^T(\alpha)\right]^{-1}\left(B_l^T(\alpha)B_l(\alpha)\right)^{-1/2}$$

and

$$C(\alpha) = \begin{bmatrix} D(\alpha) & B^{\perp}(\alpha) \end{bmatrix}.$$

One has that multiplying (16) by left by $C^T(\alpha)$ and by right by $C(\alpha)$ yields

$$\begin{bmatrix} D^T(\alpha)Q(\alpha)D(\alpha) - \mu(\alpha)I & D^T(\alpha)Q(\alpha)B^{\perp}(\alpha) \\ B^{\perp T}(\alpha)Q(\alpha)D(\alpha) & B^{\perp T}(\alpha)Q(\alpha)B^{\perp}(\alpha) \end{bmatrix}. \tag{19}$$

If the matrices were parameter independent, it would be possible to take a sufficiently large μ such that (19) is negative definite implying that (16) is also negative definite [15]. However, in the parameter dependent case, $\mu(\alpha)$ is an arbitrary function and there is no more guarantee that exists a sufficiently large $\mu(\alpha)$ such that (19) is negative definite.

Nevertheless, under the extra assumption that $Q(\alpha)$ is a continuous function and $B_l(\alpha)$ and $B_r(\alpha)$ can be chosen as a continuous functions, then using that Δ_N is a compact set it is possible to prove that there exists an $\ell \in \mathbb{R}$ such that $D^T(\alpha)Q(\alpha)D(\alpha) \prec \ell I$ for all $\alpha \in \Delta_N$. Taking $\mu > \ell$ sufficiently great, it now holds that (19) is negative definite.

Proceeding as the previous example, if Q and B satisfies the hypothesis of one of the results of the preceding section, it is possible to choose without loss of generality a constant $\mu(\alpha) = \bar{\mu}$.

Taking

$$F(\alpha) = -\frac{\bar{\mu}}{2}A^T(\alpha),\ G(\alpha) = \frac{\bar{\mu}}{2}I,$$

then without loss of generality, $G(\alpha)$ can be assumed as constant and $F(\alpha)$ can be assumed as a homogeneous polynomial of degree 1. Thus, in this case, there would be no gain in imposing a higher order homogeneous polynomial, or any other more complex structure for $F(\alpha)$.

IV. Conclusion

In this paper it is proposed a set of sufficient conditions which enables the uniformity of the extra variable introduced by the application of Finsler's lemma in the context of parameter dependent matrices. Using these conditions it is possible to show that nothing is gained imposing additional structures in the extra scalar variable, since one can always choose it as a constant variable for all parameters of the involved dependent matrices.

Acknowledgements

The authors would like to thanks Eduardo S. Tognetti and Ricardo C. L. F. Oliveira for the discussions relevant to this paper and also the anonymous reviewers for their comments. We would also like to thanks the Brazilian agencies CNPq and CAPES which partially supported this work, and the IEEE CSS for the travel support to the first author.

References

[1] A. M. Lyapunov, "The general problem of the stability of motion," *Int. J. Control*, vol. 55, no. 3, pp. 531–534, 1992.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM Studies in Applied Mathematics, 1994.

[3] L. El Ghaoui and S. I. Niculescu, eds., *Advances in Linear Matrix Inequality Methods in Control*. Advances in Design and Control, Philadelphia, PA: SIAM, 2000.

[4] G. R. Duan and H. H. Yu, *LMIs in Control Systems: Analysis, Design and Applications*. CRC Press, 2013.

[5] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox User's Guide*. Natick, MA: The Math Works, 1995.

[6] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Method Softw.*, vol. 11, no. 1–4, pp. 625–653, 1999. `http://sedumi.ie.lehigh.edu/`.

[7] K. C. Toh, M. J. Todd, and R. Tütüncü, "SDPT3 — A Matlab software package for semidefinite programming, Version 1.3," *Optim. Method Softw.*, vol. 11, no. 1, pp. 545–581, 1999.

[8] B. Borchers, "CSDP, A C library for semidefinite programming," *Optim. Method Softw.*, vol. 11, pp. 613–623, 1999.

[9] M. Yamashita, K. Fujisawa, and M. Kojima, "Implementation and evaluation of SDPA 6.0 (Semidefinite Programming Algorithm 6.0)," *Optim. Method Softw.*, vol. 18, pp. 491–505, August 2003.

[10] J. S. Shamma, "An overview of LPV systems," in *Control of linear parameter varying systems with applications*, pp. 3–26, Springer, 2012.

[11] A. Ben-Tal and A. Nemirovski, "Robust convex optimization," *Mathematics of Operations Research*, vol. 23, no. 4, pp. 769–805, 1998.

[12] P. Apkarian and H. D. Tuan, "Parametrized LMIs in control theory," *SIAM J. Control Optim.*, vol. 38, pp. 1241–1264, May 2000.

[13] P.-A. Bliman, "An existence result for polynomial solutions of parameter-dependent LMIs," *Syst. Control Letts.*, vol. 51, pp. 165–169, Mar. 2004.

[14] P.-A. Bliman, R. C. L. F. Oliveira, V. F. Montagner, and P. L. D. Peres, "Existence of homogeneous polynomial solutions for parameter-dependent linear matrix inequalities with parameters in the simplex," in *Proc. 45th IEEE Conf. Decision Control*, (San Diego, CA, USA), pp. 1486–1491, Dec. 2006.

[15] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control* (S. O. Reza Moheimani, ed.), vol. 268 of *Lecture Notes in Control and Information Science*, pp. 241–257, New York, NY: Springer-Verlag, 2001.

[16] R. C. L. F. Oliveira and P. L. D. Peres, "Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations," *IEEE Trans. Autom. Control*, vol. 52, pp. 1334–1340, July 2007.

[17] V. F. Montagner, R. C. L. F. Oliveira, and P. L. D. Peres, "Convergent LMI relaxations for quadratic stabilizability and $\mathscr{H}_\infty$ control for Takagi–Sugeno fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, pp. 863–873, August 2009.

[18] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*. Communications and Control Engineering, Springer, 2015.

[19] P. Finsler, "Über das vorkommen definiter und semidefiniter formen in scharen quadratischer formen," *Comment. Math. Helv.*, vol. 9, pp. 188–192, 1937.

[20] Y. Zi-Zong and G. Jin-Hai, "Some equivalent results with Yakubovich's S-lemma," *SIAM J. Control Optim.*, vol. 48, no. 7, pp. 4474–4480, 2010.

[21] K. J. Arrow, F. J. Gould, and S. M. Howe, "A general saddle point result for constrained optimization," *Math. Program.*, vol. 5, no. 1, pp. 225–234, 1973.

[22] F. Uhlig, "A recurring theorem about pairs of quadratic forms and extensions: A survey," *Lin. Alg. Appl.*, vol. 25, no. 1, pp. 219–237, 1979.

[23] C. Hamburger, "Two extensions to Finsler's recurring theorem," *Appl. Math. Optim.*, vol. 40, no. 2, pp. 183–190, 1999.

[24] I. Pólik and T. Terlaky, "A survey of the S-lemma," *SIAM Rev.*, vol. 49, no. 3, pp. 371–418, 2007.

[25] J. Cimprič, "Finsler's lemma for matrix polynomials," *Lin. Alg. Appl.*, vol. 465, pp. 239–261, 2015.