Published in 2015 54th IEEE Conference on Decision and Control (CDC). DOI:10.1109/CDC.2015.7403158

# Finite time boundedness and stability analysis of discrete time uncertain systems

H. T. M. Kussaba, J. Y. Ishihara and R. A. Borges

*Abstract*— **In this paper the problems of finite time boundedness and stability of uncertain systems in discrete-time are addressed. By using the Finsler lemma, novel parameter dependent linear matrix inequalities analysis conditions are designed. These conditions can be efficiently solved using a homogeneous polynomial relaxation procedure and readily available numerical solvers.**

**In the numerical examples section, it is shown that the proposed finite time conditions are less conservative than others linear matrix inequality based conditions proposed in the literature.**

*Index Terms*— **finite time stability, finite time boundedness, linear matrix inequality, uncertain systems**

## I. Introduction

The concept of finite time stability (FTS) was introduced almost fifty years ago [1]–[3] in order to characterize system trajectories during a finite time horizon. In contrast to Lyapunov stability theory, where stability is only a qualitative property of the system, the finite time stability concept considers qualitative and precise quantitative properties. More specifically, finite time stability characterizes whether the trajectories of the system will be restricted to some prescribed constraints during a finite horizon run of the system. The characterization of system trajectories during transient time provides a good framework to design control strategies able to avoid excitation of nonlinear dynamics, deal with systems with saturation, and so on [4]. Also, in order to guarantee these prescribed constraints in the presence of external disturbances, the concept of finite time boundedness (FTB) was recently introduced in [5].

For linear systems, there are necessary and sufficient FTS conditions based on differential and difference linear matrix inequalities (DLMIs) and based on the state transition matrix of the system [6]–[8]. The FTB analysis is more complex and, to the best of the authors knowledge, a FTB necessary and sufficient condition exists only for continuous time systems [8], [9]. However, since these necessary and sufficient conditions are in general computationally intractable, most of the finite time results focused on FTS/FTB sufficient analysis based on linear matrix inequalities (LMIs) [10]–[12].

In this paper, the problem of FTS/FTB analysis of discrete time uncertain systems is addressed. Uncertainties in the parameters of the system are very common in real scenarios and in many situations they are not negligible and must be properly addressed in order to enable a feasible controller design. For discrete time systems, while in the context of Lyapunov stability several works have been done regarding system uncertainties (to name a few examples, [13]–[16]), in the finite time context few works deal with FTS/FTB of uncertain systems [12], [17], [18]. In particular, [18] deals with nonlinear systems with conic-type nonlinearities and [12] with singular systems with Markovian jumps. Differently from [12], [17], [18] which use norm bounded matrix uncertainties together with the Petersen lemma [19], the uncertainty of the system in this work will be described using a polytopic representation. In comparison with the norm-bounded uncertainty representation, the polytopic representation is more flexible and describes more accurately the uncertainty [20, p. 335]. In this context, the polytopic representation considered in this paper is dealt with the Finsler's lemma, Pólya's theorem and homogeneous polynomial relaxation.

The main contributions of this paper are as following. To derive the proposed FTS/FTB analysis condition, an approach based on [21] is used to apply a version of Finsler's lemma [22] to a difference inequality from which stems the FTS/FTB conditions. This approach is new even in the scenario that the system matrices are precisely known and provides less conservative analysis conditions when compared with those appeared in the literature in the context of LMIs. As mentioned before, there is no necessary and sufficient characterization for FTB of uncertain discrete-time systems in the literature, and the proposed conditions, albeit only sufficient, reduce the conservatism when compared to the ones from the literature.

This paper is organized as follows. In Section II, is established the problem of FTS and FTB robustness against parameter uncertainty in a polytopic region. In Section III a new LMI analysis condition based on Finsler's lemma is derived. Numerical examples are shown in Section IV and finally, the conclusion is presented in Section V.

In the sequel the following notation will be used: $\mathbb{Z}_+$ is the set of non-negative integer numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{m\times n}$ is the set of real matrices of order $m\times n$, $\mathbb{S}^n$ is the set of symmetric real matrices of order $n\times n$, $\mathbb{S}^n_+$ is the set of symmetric positive definite matrices of order $n\times n$. $P\succ 0$ indicates that $P$ is symmetric positive definite matrix. The term $\star$ indicates symmetric terms in symmetric matrices. $\lambda_{max}(A)$ and $\lambda_{min}(A)$ are respectively the maximum and the minimum eigenvalue of the matrix $A$. $A^T$ denotes the transpose of matrix $A$. The matrix $\mathbf{I}_n$ denotes the identity matrix of order $n$. The notation $\mathrm{He}\{A\}$ indicates $A+A^T$.

The authors are with the Department of Electrical Engineering, University of Brasília – UnB, 70910-900, Brasília, DF, Brazil. Currently, J. Y. Ishihara is a visiting associate researcher at Univ. California Los Angeles (UCLA). kussaba@lara.unb.br, ishihara@ene.unb.br, raborges@ene.unb.br

## II. Problem statement and preliminary results

Consider a linear system with $k \in \{1,\ldots,N_f\}$ given by

$$x(k+1) = Ax(k) + G\omega(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state space vector, $\omega(k) \in \mathbb{R}^r$ is a noise input and $A$ and $G$ are real matrices with appropriate dimensions. The concept of FTS is formalized in Definition 1.

**Definition 1.** System (1) with $G = 0$ is **finite time stable** with respect to $(c_1, c_2, R, N_f)$, with $c_2 > c_1 \geq 0$, $N_f \in \mathbb{Z}_+$ and $R \succ 0$ if for any solution $x$ of (2) in the time horizon $1 \leq k \leq N_f$ one has that

$$x^T(0)Rx(0) \leq c_1 \implies x^T(k)Rx(k) < c_2, \forall k \in \{1,\ldots,N_f\}.$$

The geometric meaning of Definition 1 is illustrated in Figure 1 for solutions $x$, $y$ and $z$ of a system: the pair $(c_2, R)$ defines a ellipsoid restraining all the trajectories of (1) starting in the ellipsoid defined by the pair $(c_1, R)$ during $k = 1,\ldots,N_f$.

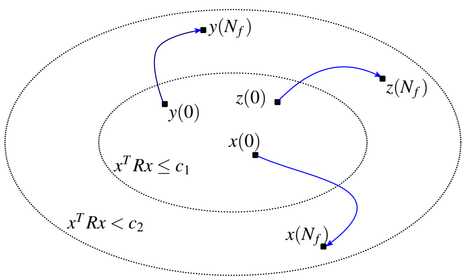


Fig. 1. Ellipsoids bounding the trajectories of a system.

The concept of FTB, which extends the FTS to deal with external disturbances, is formalized in Definition 2.

**Definition 2.** System (1) is **finite time bounded** with respect to $(c_1, d, c_2, R, N_f)$, with $c_2 > c_1 \geq 0$, $d \geq 0$, $N_f \in \mathbb{Z}_+$ and $R \succ 0$ if for any solution $x$ of (2) in the time horizon $1 \leq k \leq N_f$ one has that

$$x^T(0)Rx(0) \leq c_1 \implies x^T(k)Rx(k) < c_2, \quad \begin{array}{c} \forall k \in \{1,\ldots,N_f\}, \\ \forall \omega \in \mathscr{W}_d, \end{array}$$

where $\mathscr{W}_d = \{\omega(\cdot) : \mathbb{Z}_+ \to \mathbb{R} \mid \omega^T(k)\omega(k) \leq d, k = 0,\ldots N_f\}$.

The presence of uncertainties in (1) can be modeled with parametric uncertainties in the system matrices. In this case, the system to be handled is given by

$$x(k+1) = A(\alpha(k))x(k) + G(\alpha(k))\omega(k), \tag{2}$$

where $\alpha(k) \in \mathbb{R}^N$ is a vector of uncertain parameters belonging to the unit $N$-simplex

$$\Delta^N = \left\{ \theta \in \mathbb{R}^N : \sum_{i=1}^{N} \theta_i = 1, \ \theta_i \geq 0 \right\}.$$

The matrices $A(\alpha(k))$ and $G(\alpha(k))$ depends linearly on the uncertainty $\alpha(k) = (\alpha_1(k),\ldots,\alpha_N(k))$, viz.:

$$A(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) A_i, \ G(\alpha(k)) = \sum_{i=1}^{N} \alpha_i(k) G_i,$$

where $A_i, B_i$, $i = 1,\ldots,N$ are fixed, non-dependent of $\alpha(k)$, known matrices called the **vertices** of $A(\alpha(k))$ and $B(\alpha(k))$, respectively.

The concept of robust FTS and FTB is formalized next in definitions 3 and 4.

**Definition 3.** System (2) with $G = 0$ is robustly FTS (RFTS) with respect to $(c_1, c_2, R, N_f)$, with $c_2 > c_1 \geq 0$, $N_f \in \mathbb{Z}_+$ and $R \succ 0$ if for any solution $x$ of (2) in the time horizon $1 \leq k \leq N_f$ one has that

$$x^T(0)Rx(0) \leq c_1 \implies x^T(k)Rx(k) < c_2, \quad \begin{array}{c} \forall k \in \{1,\ldots,N_f\}, \\ \forall \alpha(k) \in \Delta^N. \end{array}$$

**Definition 4.** System (2) is robustly FTB (RFTB) with respect to $(c_1, d, c_2, R, N_f)$, with $c_2 > c_1 \geq 0$, $d \geq 0$, $N_f \in \mathbb{Z}_+$ and $R \succ 0$ if for any solution $x$ of (2) in the time horizon $1 \leq k \leq N_f$ one has that

$$x^T(0)Rx(0) \leq c_1 \implies x^T(k)Rx(k) < c_2, \quad \begin{array}{c} \forall k \in \{1,\ldots,N_f\}, \\ \forall \omega \in \mathscr{W}_d, \\ \forall \alpha(k) \in \Delta^N. \end{array}$$

where $\mathscr{W}_d = \{\omega(\cdot) : \mathbb{Z}_+ \to \mathbb{R} \mid \omega^T(k)\omega(k) \leq d, k = 0,\ldots N_f\}$.

In particular, if $d = 0$, the external signal is identically equal to zero and the RFTB notion of Definition 4 reduces to the RFTS notion of Definition 3. It is also important to remark that the definition of FTB used in this paper and in [17] is different from the definition used in [23]. In [23], only the initial condition of the external signal is assumed to be bounded while in [17] and in this paper the noise is assumed to be bounded during all the finite time horizon. Although this is a more restrictive assumption, the dynamics of the external signal in [23] is precisely known while in [17] and in this paper no assumptions besides the boundedness is done about the dynamics of the noise.

To derive a novel analysis condition for (2) be RFTS/RFTB, Finsler's lemma will be used. It is important to remark that the version of Finsler's lemma stated in this work is from [21] and it is slight different from the statement of the original lemma proved in [22].

**Lemma 1** (Finsler's lemma)**.** *[21], [22] Let $x \in \mathbb{R}^n$, $\mathscr{Q} \in \mathbb{S}^n$ and $\mathscr{B} \in \mathbb{R}^{m \times n}$ such that rank$(\mathscr{B}) = r < n$. Then the following statements are equivalent:*

1) $x^T \mathscr{Q} x < 0 \ \forall x \in \mathbb{R}^n$ *such that* $x \neq 0$ *and* $\mathscr{B}x = 0$,
2) $\exists \mathscr{X} \in \mathbb{R}^{n \times m} : \mathscr{Q} + \mathscr{X}\mathscr{B} + \mathscr{B}^T\mathscr{X}^T \prec 0.$

The matrix $\mathscr{X}$ of the second statement of Lemma (1) can be seen as a matrix Lagrange multiplier [21] taking into account the linear constraint originating from $\mathscr{B}x = 0$. By rewriting the system equation and a difference inequality as item 1) of Lemma 1 it will be possible to derive an analysis condition with a search space enlarged by the extra variable $\mathscr{X}$ of item 2) of Lemma 1. This extra variable, as will be shown in the numeric examples, will allow to obtain less conservative LMI conditions even when dealing with precisely known matrices.

Along Lemma 1, homogeneous polynomial relaxation [24] and Pólya's theorem version for homogeneous polynomial matrices [25] will also be used to deal with parameter dependent LMI (PD-LMI) that stems when treating the case that there is uncertainty in the system matrices. To use these relaxation techniques, the concept of a homogeneous polynomially parameter-dependent (HPPD) matrix is briefly reviewed first.

**Definition 5.** A matrix valued function $M : \Delta^N \to \mathbb{R}^{m\times n}$ is **homogeneous polynomially parameter-dependent** (HPPD) on $\alpha \in \Delta^N$ with degree $g$ if it can be expressed as

$$M(\alpha) = \sum_{p\in\mathscr{S}_g} \alpha_1^{p_1}\alpha_2^{p_2}\cdots\alpha_N^{p_N} M_p, \tag{3}$$

with

$$\mathscr{S}_g = \left\{ p = (p_1,\ldots,p_N) : p_i \in \mathbb{Z}_+, i = 1,\ldots,N; \sum_{i=1}^{N} p_i = g \right\}.$$

The matrices $M_p$ are called the **matrices coefficients** of the monomials of $M(\alpha)$.

The importance of HPPD matrices follows from the work presented in [26], where it is proved that if a generic LMI dependent on a parameter on $\Delta^N$ has a solution, then without loss of generality, there will also exists a parameter dependent solution which is a HPPD matrix. This motivates the following procedure to convert a parameter dependent LMI on parameter independent LMIs. By replacing the variables of the PD-LMI by HPPD matrices and imposing the matrices coefficients (that are all parameter independent) to be of the same sign of the original PD-LMI, new parameter independent LMIs can be obtained using the matrices coefficients of the HPPD matrices. Those parameter independent LMIs are such that their solution set is included in the original PD-LMI.

It is important to note that it is not necessary that the coefficients of a HPPD matrix defined on $\Delta^N$ be negative (resp. positive) for it to be negative (resp. positive). This motivates the use of a matrix version of Pólya's theorem [25]: if an HPPD matrix $\mathscr{F}(\alpha)$ is positive on $\Delta^N$, then there will exist a scalar $f$ sufficiently large such that all the coefficients of the HPPD matrix

$$(\alpha_1 + \cdots + \alpha_N)^f \mathscr{F}(\alpha)$$

are also positive. This gives another relaxation procedure to verify if a parameter dependent LMI is positive or negative, that is, multiply the original PD-LMI by $\sum_{i=1}^{N}\alpha_i$ in order to eventually obtain a HPPD matrix with positive coefficients. For more details of these procedures the reader is referred to [24].

It is important to remark that although these LMI relaxation procedures are systematic, they can turn very complex as $f$ and $g$ increases. In spite of that, the specialized parser ROLMIP[1] can be used to automatically carry this relaxation [27].

[1]Available for download at `http://www.dt.fee.unicamp.br/~agulhari/rolmip/rolmip.htm`.

## III. MAIN RESULTS

The following theorems state the proposed LMI conditions based on Finsler's lemma. In Theorem 1 are stated the conditions for FTB and in Corollary 1 are stated the conditions for FTS.

**Theorem 1.** *System (2) is RFTB with respect to $(c_1, d, c_2, R, N_f)$ if there exist matrices $P_1 \in \mathbb{S}_+^n$, $P_2 \in \mathbb{S}_+^r$; matrices valued functions $X_1(\alpha) \in \mathbb{R}^{n\times n}$, $X_2(\alpha) \in \mathbb{R}^{n\times n}$, $X_3(\alpha) \in \mathbb{R}^{r\times n}$; and positive scalars $\lambda_1, \lambda_2$, $\lambda_3$ and $\gamma > 1$ such that for any $\alpha \in \Delta^N$*

$$\begin{gathered}\begin{bmatrix} \mathscr{M}_{11}(\alpha) & \mathscr{M}_{12}(\alpha) & -X_1(\alpha)G(\alpha) + X_3^T(\alpha) \\ \star & \mathscr{M}_{22}(\alpha) & -X_2(\alpha)G(\alpha) - A^T(\alpha)X_3^T(\alpha) \\ \star & \star & -\gamma P_2 - \mathrm{He}\{X_3(\alpha)G(\alpha)\} \end{bmatrix} \prec 0,\\ \mathscr{M}_{11}(\alpha) = P_1 + \mathrm{He}\{X_1(\alpha)\},\\ \mathscr{M}_{12}(\alpha) = -X_1(\alpha)A(\alpha) + X_2^T(\alpha),\\ \mathscr{M}_{22}(\alpha) = -\gamma P_1 - \mathrm{He}\{X_2(\alpha)A(\alpha)\},\end{gathered} \tag{4}$$

$$\lambda_3 R \prec P_1 \prec \lambda_1 R, \tag{5}$$

$$P_2 \prec \lambda_2 \mathbf{I}_r, \tag{6}$$

$$\lambda_1 c_1 + \lambda_2 d \frac{\left(1 - \frac{1}{\gamma^{N_f}}\right)}{\left(1 - \frac{1}{\gamma}\right)} < \lambda_3 \frac{c_2}{\gamma^{N_f}}. \tag{7}$$

*Proof:* Consider the Lyapunov function candidate given by

$$V(x(k)) = x^T(k)P_1x(k) \tag{8}$$

and define $\tilde{P}_1 = R^{-1/2}P_1R^{-1/2}$.

Proceeding as in [17, Lemma 3.2] one can show that if the inequalities

$$V(x(k+1)) < \gamma V(x(k)) + \gamma\omega^T(k)P_2\omega(k) \tag{9}$$

and

$$\lambda_{max}(\tilde{P}_1)c_1 + \lambda_{max}(P_2)d \frac{\left(1 - \frac{1}{\gamma^{N_f}}\right)}{\left(1 - \frac{1}{\gamma}\right)} < \frac{c_2\lambda_{min}(\tilde{P}_1)}{\gamma^{N_f}} \tag{10}$$

are satisfied, then the system (2) is RFTB with respect to $(c_1, d, c_2, R, N_f)$.

In fact, using Grönwall's lemma one has that (9) implies that

$$\begin{aligned} V(x(k)) &\le \gamma^k V(x(0)) + \sum_{i=1}^{k} \gamma^i \omega^T(k-i)P_2\omega(k-i) \\ &\le \gamma^k x^T(0)P_1x(0) + \lambda_{max}(P_2)d\frac{\gamma^{k+1}-\gamma}{\gamma-1} \\ &\le \gamma^k \lambda_{max}(\tilde{P}_1)c_1 + \lambda_{max}(P_2)d\frac{\gamma^{k+1}-\gamma}{\gamma-1} \\ &\le \gamma^N \left(\lambda_{max}(\tilde{P}_1)c_1 + \lambda_{max}(P_2)d\frac{1-\frac{1}{\gamma^N}}{1-\frac{1}{\gamma}}\right). \end{aligned}$$

On the other hand, one has that

$$V(x(k)) \ge \lambda_{min}(\tilde{P}_1)x^T(k)Rx(k),$$

and thus

$$x^T(k)Rx(k) < \frac{\gamma^N}{\lambda_{min}(\tilde{P}_1)}\left(\lambda_{max}(\tilde{P}_1)c_1 + \lambda_{max}(P_2)d\frac{1-\frac{1}{\gamma^N}}{1-\frac{1}{\gamma}}\right),$$

which with (10) implies that system (2) is RFTB with respect to $(c_1, d, c_2, R, N_f)$.

Using (8), we have that (9) can be written as $\bar{x}^T \mathscr{Q} \bar{x} < 0$, where

$$\mathscr{Q} = \begin{bmatrix} P_1 & 0 & 0 \\ \star & -\gamma P_1 & 0 \\ \star & \star & -\gamma P_2 \end{bmatrix}, \bar{x} = \begin{bmatrix} x(k+1) \\ x(k) \\ \omega(k) \end{bmatrix}.$$

Defining

$$\mathscr{B}(\alpha) = \begin{bmatrix} \mathbf{I}_n & -A(\alpha) & -G(\alpha) \end{bmatrix}, \mathscr{X}(\alpha) = \begin{bmatrix} X_1(\alpha) \\ X_2(\alpha) \\ X_3(\alpha) \end{bmatrix},$$

Lemma 1 can be applied pointwise to each $\alpha \in \Delta^N$ to yield (4).

Finally, by imposing conditions (5)-(7), one has that (10) is satisfied. ■

A simple corollary that yields a sufficient condition to analysis the RFTS of (2) is given next. Albeit being only a sufficient condition, Corollary 1 gives a computationally faster alternative to the DLMI [8] or singular value based computations [23, Rem. 2].

**Corollary 1.** *System (2) is RFTS with respect to $(c_1, c_2, R, N_f)$ if there exists $P \in \mathbb{S}^n_+$; matrices valued functions $X_1(\alpha) \in \mathbb{R}^{n\times n}$, $X_2(\alpha) \in \mathbb{R}^{n\times n}$; and positive scalars $\lambda_1$, $\lambda_3$ and $\gamma > 1$ such that for any $\alpha \in \Delta^N$*

$$\begin{bmatrix} P + \mathrm{He}\{X_1(\alpha)\} & -X_1(\alpha)A(\alpha) + X_2^T(\alpha) \\ \star & -\gamma P - \mathrm{He}\{X_2(\alpha)A(\alpha)\} \end{bmatrix} \prec 0, \quad (11)$$

$$\lambda_3 R \prec P \prec \lambda_1 R, \quad (12)$$

$$\lambda_1 c_1 < \lambda_3 \frac{c_2}{\gamma^{N_f}}. \quad (13)$$

*Proof:* Similar to Theorem 1. ■

It is important to notice that both Theorem 1 and Corollary 1 lead to PD-LMI feasibility problems, that is LMIs that must be satisfied for all parameters $\alpha \in \Delta^N$. Although this is a problem of infinite dimension in the parameter $\alpha$, the relaxation schemes of Section II can be used. By supposing that the extra variables $X_i(\alpha)$, $i = 1, \ldots, 3$ introduced by Finsler's lemma are HPPD matrices, it is possible to find sufficient LMI conditions written only in terms of the vertices of the matrices $A$ and $G$ and the coefficients of the HPPD matrices $X_i(\alpha)$. As the level of relaxation increases, it is possible to achieve less conservative sets of conditions.

Another important remark is that if the system matrices are precisely known, the PD-LMIs relapses to LMIs again and no conservatism is introduced by supposing that the matrices valued functions $X_i(\alpha)$, $i = 1, \ldots, 3$ are constant — which corresponds to using a HPPD matrix with degree $g = 0$.

Besides the feasibility problems treated in Theorem 1 and Corollary 1, it is also interesting to consider optimization problems constrained to the conditions of Theorem 1 and Corollary 1. For instance, one may be interested in computing the maximum $d$ to analyze how far the system can reject disturbances or computing the minimum $c_2$ to analyze how far the trajectories of the system can be restrained. Firstly, we treat the problem of minimization of $c_2$ in Theorem 2.

**Theorem 2.** *For given $g \in \mathbb{Z}_+$ and $\bar{\lambda}_3 > 0$, let $c_2^*(g)$ be an optimal solution of* min $c_2$ *subject to (4)-(7) with $\lambda_3 = \bar{\lambda}_3$ and $X_i(\alpha)$, $i = 1, \ldots, 3$ being HPPD matrices taking values on $\alpha \in \Delta^N$ with degree $g$. Then one has that $c_2^*(g+1) \le c_2^*(g)$.*

*Proof:* If there exists positive scalars $\lambda_1$, $\lambda_2$, $\bar{\lambda}_3$ and $\gamma > 1$; positive definite matrices $P_1 \in \mathbb{S}^n_+$, $P_2 \in \mathbb{S}^r_+$; and HPPD matrices $X_1(\alpha) \in \mathbb{R}^{n\times n}$, $X_2(\alpha) \in \mathbb{R}^{n\times n}$, $X_3(\alpha) \in \mathbb{R}^{r\times n}$ with degree $g$ such that (4)-(7) hold, then, since $\alpha \in \Delta^N$, the following HPPD matrices with degree $g+1$

$$\left(\sum_{i=1}^{N} \alpha_i\right) X_j(\alpha), \; j = 1, \ldots, 3,$$

and $\lambda_1$,$\lambda_2$, $\bar{\lambda}_3$, $\gamma$, $P_1$, $P_2$ are also a particular solution of (4)-(7). Hence the minimization of $c_2$ subject to (4)-(7) for $g+1$ produces at least the same optimal value obtained with $g$, which implies that $c_2(g+1) \le c_2(g)$. ■

The problems of maximization of $c_1$ and $d$ are also similar and it is treated next in Theorem 3.

**Theorem 3.** *For given $g \in \mathbb{Z}_+$ and $\bar{\lambda}_1, \bar{\lambda}_2 > 0$, let $c_1^*(g)$ (resp. $d^*(g)$) be an optimal solution of* max $c_1$ *(resp.* max $d$*) subject to (4)-(7) with $\lambda_1 = \bar{\lambda}_1$ (resp. $\lambda_2 = \bar{\lambda}_2$) and $X_i(\alpha)$, $i = 1, \ldots, 3$ being HPPD matrices taking values on $\alpha \in \Delta^N$ with degree $g$.*

*Then one has that $c_1^*(g+1) \ge c_1^*(g)$ and that $d^*(g+1) \ge d^*(g)$.*

*Proof:* Similar to Theorem 2. ■

It should be noted that increasing the degree $g$ of a HPPD matrix in these relaxation scheme increases the number of decision variables. By using Pólya's relaxation mentioned in Section II, however, it is possible to also decrease the conservatism of the LMI conditions increasing the degree of relaxation $f$ without expanding the number of decision variables (although there is a increase in the number of the LMI scalar rows).

## IV. Numerical examples

The numerical examples were performed using the SDP solver SeDuMi [28] and the parsers YALMIP [29] and ROLMIP [27] within MATLAB environment.

**Example 1.** To show that by using Finsler's lemma alone, without uncertainty parameters, the conservatism is decreased in comparison with the FTB analysis conditions of [23] and [17] let consider precisely known matrices in this example, that is, we consider that $\alpha = 1$, $N = 1$ and $A = A_1$. The matrices of (2) are chosen as

$$A = \begin{bmatrix} -1.20 & -0.09 & 0.02 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \end{bmatrix},$$

$$G = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.5 & 0.5 \end{bmatrix},$$

and the FTB parameters to be considered are $c_1 = 1$, $c_2 = 18$, $d = 1.1$, $N_f = 5$ and $R = \mathbf{I}_3$.

To compare with the FTB analysis condition of [23], let consider an external sinusoidal signal given by

$$\omega(k+1) = F\omega(k),$$

where

$$F = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}.$$

Since $F$ has complex eigenvalues of unitary modulus, it is easy to prove that $\omega^T(0)\omega(0) = \omega^T(k)\omega(k) \leq d$ for all $k = 1, \ldots, N_f$. Thus, $\omega$ belongs to the noise class that is tackled in Definition 4.

Although using Lemma 3.2 from [17] with $H = 0$ and $E_1 = 0$ and using Lemma 1 from [23] it is not possible to known if the system is finite time bounded, the use of Theorem 1 assures the FTB of the system.

To graphically illustrate the FTB of the system, a time-simulation of 100 random initial conditions whose norm is less than or equal to $c_1 = 1$ was performed and plotted in Figure 2. As can be seen, every trajectory maintains its norm below than $c_2 = 18$ during the interval $k = 1, \ldots, N_f$.

It can also be seen that the maximum norm obtained by the trajectories of this time-simulation was approximately 83% of $c_2$, indicating the small conservatism of the proposed LMI conditions.

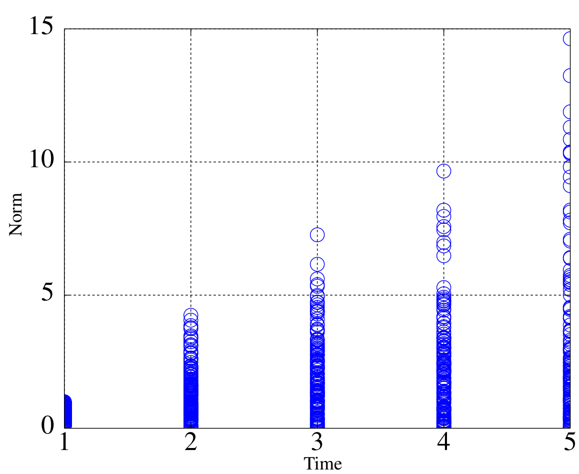


Fig. 2. Time simulation of system of Example 1.

**Example 2.** In this example, let consider that matrices $A$ and $G$ are uncertain in order to compare the proposed FTB analysis condition with the condition appeared in [17] for uncertain linear systems.

To this end, consider the system

$$x(k+1) = (A + \Delta A)x(k) + G\omega(k), \tag{14}$$

where

$$A = \begin{bmatrix} -1.00 & 0.01 & 0.01 \\ 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \end{bmatrix}, \; G = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and $\Delta A$ is an unknown matrix representing parameters uncertainties given by $\Delta A = HFE$, where

$$E = \begin{bmatrix} 0.0 & 0.1 & 0.1 \end{bmatrix}, H = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}$$

and $F$ is an uncertain scalar satisfying $-1 \leq F \leq 1$. System (14) can also be written in the polytopic form (2) taking $N = 2$ and

$$A_1 = A - HE_1, \; A_2 = A + HE_1.$$

The goal is to find the minimum value of $c_2$ that guarantees that (14) is FTB for $c_1 = 0$, $d = 1$, $N_f = 4$ and $R = \mathbf{I}_3$. In this framework both Lemma 3.2 from [17] and Theorem 2 was applied in the search of $c_2$. By using Lemma 3.2 from [17] was not possible to find the minimum $c_2$ that guarantees that (14) is FTB. On the other hand, by using Theorem 2 with $\bar{\lambda}_3 = 1$, $g = 2$ and $f = 1$ it was possible to find that for all $c_2 > 34.64$ the system is guaranteed to be FTB, what illustrates the effectiveness of the proposed method.

## V. Conclusion

In this paper, a novel LMI-based condition for the FTS and FTB analysis of a uncertain polytopic discrete-time system was derived by the use of Finsler's lemma. The numerical examples show that the proposed analysis condition is less conservative than others LMI based analysis conditions presented in the literature, even if the matrices of the system are precisely known.

Although only analysis conditions are presented in this paper, further work should be done to use these conditions for controller and filter synthesis problems.

## Acknowledgments


The authors would like to thanks the Brazilian agencies CNPq and CAPES which partially supported this work, and the IEEE CSS for the travel support to the first author.


## References


[1] G. Kamenkov, "On stability of motion over a finite interval of time," *J. Appl. Math. Mech.*, vol. 17, pp. 529–540, 1953.

[2] A. Lebedev, "On stability of motion during a given interval of time," *J. Appl. Math. Mech.*, vol. 18, pp. 139–148, 1954.

[3] A. Lebedev, "The problem of stability in a finite interval of time," *J. Appl. Math. Mech.*, vol. 18, pp. 75–94, 1954.

[4] F. Amato, M. Ariola, and C. Cosentino, "Finite-time stability of linear time-varying systems: analysis and controller design," *IEEE Trans. Autom. Control*, vol. 55, pp. 1003–1008, Apr. 2010.

[5] F. Amato, M. Ariola, and P. Dorato, "Finite-time control of linear systems subject to parametric uncertainties and disturbances," *Automatica*, vol. 37, pp. 1459–1563, Sept. 2001.

[6] F. Amato, M. Carbone, M. Ariola, and C. Cosentino, "Finite-time stability of discrete-time systems," in *Proc. 2004 Amer. Control Conf.*, vol. 2, pp. 1440–1444, July 2004.

[7] F. Amato, M. Ariola, and C. Cosentino, "Finite-time control of discrete-time linear systems: Analysis and design conditions," *Automatica*, vol. 46, pp. 919–924, May 2010.

[8] F. Amato, R. Ambrosino, M. Ariola, C. Cosentino, and G. De Tommasi, *Finite-Time Stability and Control*. London, UK: Springer, 2014.

[9] H. Ichihara and H. Katayama, "Necessary and sufficient conditions for finite-time boundedness of linear continuous-time systems," in *Proc. 48th IEEE Conf. Decision Control — 28th Chinese Control Conf.*, pp. 3214–3219, Dec. 2009.

[10] F. Chen, S. Xu, M. Zhang, and Y. Zou, "Finite-time boundedness and stabilisation for a class of non-linear quadratic time-delay systems with disturbances," *IET Control Theory & Appl.*, vol. 7, pp. 1683–1688, Sept. 2013.

[11] S. He and F. Liu, "Finite-time boundedness of uncertain time-delayed neural network with markovian jumping parameters," *Neurocomputing*, vol. 103, pp. 87–92, Mar. 2013.

[12] Y. Zhang, P. Shi, S. K. Nguang, and Y. Song, "Robust finite-time $\mathscr{H}_\infty$ control for uncertain discrete-time singular systems with Markovian jumps," *IET Control Theory & Appl.*, vol. 8, no. 12, pp. 1105–1111, 2014.

[13] J. Bernussou, J. C. Geromel, and P. L. D. Peres, "Stabilisabilité de systèmes linéaires: Tests par programmation linéaire," *Comptes Rendus de l'Académie des Sciences CRAS - Paris*, vol. 307, pp. 683–688, 1988.

[14] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. New York: Prentice Hall, 1998.

[15] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, "A new discrete-time robust stability condition," *Syst. Control Letts.*, vol. 37, pp. 261–265, July 1999.

[16] D. C. W. Ramos and P. L. D. Peres, "A less conservative LMI condition for the robust stability of discrete-time uncertain systems," *Syst. Control Letts.*, vol. 43, pp. 371–378, August 2001.

[17] L. Zhu, Y. Shen, and C. Li, "Finite-time control of discrete-time systems with time-varying exogenous disturbance," *Commun. Nonlin. Sci. Numer. Simul.*, vol. 14, pp. 361–370, Feb. 2009.

[18] M. N. ElBsat and E. E. Yaz, "Robust and resilient finite-time bounded control of discrete-time uncertain nonlinear systems," *Automatica*, vol. 49, pp. 2292–2296, July 2013.

[19] M. V. Khlebnikov and P. S. Shcherbakov, "Petersen's lemma on matrix uncertainty and its generalizations," *Autom. Remote Control*, vol. 69, no. 11, pp. 1932–1945, 2008.

[20] F. L. Lewis, L. Xie, and D. Popa, *Optimal and Robust Estimation: With an Introduction to Stochastic Control Theory*. Automation and Control Engineering, CRC Press, 2 ed., 2007.

[21] M. C. de Oliveira and R. E. Skelton, "Stability tests for constrained linear systems," in *Perspectives in Robust Control* (S. O. Reza Moheimani, ed.), vol. 268 of *Lecture Notes in Control and Information Science*, pp. 241–257, New York, NY: Springer-Verlag, 2001.

[22] P. Finsler, "Über das vorkommen definiter und semidefiniter formen in scharen quadratischer formen," *Comment. Math. Helv.*, vol. 9, pp. 188–192, 1937.

[23] F. Amato and M. Ariola, "Finite-time control of discrete-time linear systems," *IEEE Trans. Autom. Control*, vol. 50, pp. 724–729, May 2005.

[24] R. C. L. F. Oliveira and P. L. D. Peres, "Parameter-dependent LMIs in robust analysis: Characterization of homogeneous polynomially parameter-dependent solutions via LMI relaxations," *IEEE Trans. Autom. Control*, vol. 52, pp. 1334–1340, July 2007.

[25] C. W. Scherer, "Relaxations for robust linear matrix inequality problems with verifications for exactness," *SIAM J. Matrix Anal. Appl.*, vol. 27, pp. 365–395, June 2005.

[26] P.-A. Bliman, R. C. L. F. Oliveira, V. F. Montagner, and P. L. D. Peres, "Existence of homogeneous polynomial solutions for parameter-dependent linear matrix inequalities with parameters in the simplex," in *Proc. 45th IEEE Conf. Decision Control*, (San Diego, CA, USA), pp. 1486–1491, Dec. 2006.

[27] C. M. Agulhari, R. C. L. F. de Oliveira, and P. L. D. Peres, "Robust LMI Parser: a computational package to construct LMI conditions for uncertain systems," in *XIX Brazilian Conference on Automation (CBA 2012)*, (Campina Grande, PB, Brazil), pp. 2298–2305, 2012.

[28] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. Method Softw.*, vol. 11, no. 1–4, pp. 625–653, 1999. `http://sedumi.ie.lehigh.edu/`.

[29] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. 2004 IEEE Int. Symp. on Comput. Aided Control Syst. Des.*, (Taipei, Taiwan), pp. 284–289, Sept. 2004. `http://control.ee.ethz.ch/~joloef/yalmip.php`.